

Extremal metrics on toric surfaces

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Abstract

In this paper, we study the Abreu equation on toric surfaces with prescribed edge-nonvanishing scalar curvature on Delzant polytope. In particular, we prove the existence of the extremal metric when relative K-stability is assumed.

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1 Introduction

Extremal metrics, introduced by E. Calabi, have been studied intensively in the past 20 years. There are three aspects of the topic: sufficient conditions for existence, necessary conditions for existence and uniqueness of extremal metric. The necessary conditions for the existence are conjectured to be related to certain stabilities. There are many works on this aspect ([16,

24, 25, 51, 52]). The uniqueness was completed by Mabuchi ([48]) and Chen-Tian ([16, 17])

On the other hand, there has been no much progress on the existence of extremal metrics or Kähler metrics of constant scalar curvature. One reason is that the equation is highly nonlinear and of 4th order. The problem is to solve the equation under certain necessary stability conditions. It was Tian who first gave an analytic "stability" condition which is equivalent to the existence of a Kähler-Einstein metric ([52]). In [52], Tian also defined the algebro-geometric notion of K-stability. Then in [25], Donaldson generalized Tian's definition of K-stability by giving an algebro-geometric definition of the Futaki invariant and conjectured that it is equivalent to the existence of a cscK metric. The problem may become simpler if the manifold admits more symmetry. Hence, it is natural to consider the problem on toric varieties first. Since each toric manifold M^{2n} can be represented by a Delzant polytope in \mathbb{R}^n , by Abreu, Burns and Guillemin's work, the 4th-order equation can be transformed to be an equation of real convex functions on the polytope. In a sequence of papers, Donaldson initiated a program to study the extremal metrics on toric manifolds: since each toric manifold M^{2n} can be represented by a Delzant polytope in \mathbb{R}^n , by Abreu, Burns and Guillemin's work, the 4th-order equation can be transformed to be an equation of real convex functions on the polytope, which is known as the Abreu equation; Donaldson formulated K-stability for polytopes and conjectured that the stability implies the existence of the cscK metric on toric manifolds. In [25], Donaldson also proposed a stronger version of stability which we call *uniform stability* in this paper (cf. Definition 3.2). The existence of weak solutions was solved by Donaldson under the assumption of uniform stability ([25]). Note that in [14], we prove that the uniform stability is a necessary condition. On the other hand, Zhou-Zhu ([60]) introduced the notion of properness on the modified Mabuchi functional and showed the existence of weak solutions under this assumption. When the scalar curvature $K > 0$ and $n = 2$, all these conditions are equivalent. The remaining issue is to show the regularity of the weak solutions. In a sequence of papers (cf [25–27]), Donaldson solved the problem for the metrics of constant scalar curvature on toric surfaces by proving the regularity of the weak solutions.

In this paper, we show the existence for metrics of any prescribed scalar curvature on Delzant polytope (including extremal metrics) under the assumption of uniform stability. As we know, though the equation of extremal metric is on the complex manifolds, for the toric manifolds, the

equation can be reduced to a real equation on the Delzant polytope in real space. The second author with his collaborators develops a framework to study one type of 4-th order PDE's which includes the Abreu equation(cf. [10,11,35,37–41,44–46]). We call this the *real affine technique*. This is explained in [9]. The challenging part is then to study the boundary behavior of the Abreu equation near the boundary of polytope. For the boundary of polytopes, they can be thought as the interior of the complex manifold. The important issue is then to generalize the affine techniques to the complex case. In [9], we made an attempt on this direction, in particular, we obtain the estimate of Ricci curvature \mathcal{K} near the edges in terms of the bound of H (cf. Theorem 4.13). We call the technique the *complex affine technique*. The real/complex affine techniques play an important role in both [9] and this paper.

The main theorem of this paper is

Theorem 1.1 *Let (M, ω) be a compact toric surface and Δ be its Delzant polytope. Let $K \in C^\infty(\bar{\Delta})$ be an edge-nonvanishing function. If (Δ, K) is uniformly stable, then there is a smooth T^2 -invariant metric on M that solves the Abreu equation.*

For the definitions of the edge-nonvanishing function and uniform stability, readers are referred to Definition 2.6 and Definition 3.2 respectively. For any Delzant polytope Δ , there is a unique affine linear function K such that $\mathcal{L}_K(u) = 0$ for all affine linear functions u (for the definition of \mathcal{L}_K see §3.1). On the other hand, by a result of Donaldson (cf. Proposition 3.3), we know that any relative K -polystable (Δ, K) with $K > 0$ is uniformly stable. Recently, X.-J. Wang and B. Zhou could remove the condition $K > 0$ for linear functions K (see [58]). Hence, as a corollary, we get the following Theorem.

Theorem 1.2 *Let M be a compact toric surface and Δ be its Delzant polytope. Let $K \in C^\infty(\bar{\Delta})$ be an edge-nonvanishing linear function. If (M, K) is relative K -polystable then there is a smooth T^2 -invariant metric on M with scalar curvature K .*

The paper is organized as follows: in §2 and §3, we review the background and formulate the problems. In particular, in §3.2, we explain that Theorem 1.1 can be reduced to Theorem 3.7. The proof of Theorem 3.7 occupies the rest of the paper.

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2 Kähler geometry on toric surfaces

We review Kähler geometry over toric surfaces. We assume that readers are familiar with toric varieties. The purpose of this section is to introduce the notions used in this paper.

A toric manifold is a real $2n$ -dimensional Kähler manifold (M, ω) that admits an n -torus \mathbb{T}^n Hamiltonian action. Let $\tau : M \rightarrow \mathfrak{t}^*$ be the moment map; here $\mathfrak{t} \cong \mathbb{R}^n$ is the Lie algebra of \mathbb{T}^n and \mathfrak{t}^* is its dual. The image $\bar{\Delta} = \tau(M)$ is known to be a polytope ([23]). In the literature, people use Δ for the image of the moment map. However, for convenience in this paper, we always assume that Δ is an open polytope. Δ determines a fan Σ in \mathfrak{t} . The converse is not true: Σ determines Δ only up to a certain similitude. M can be reconstructed from either Δ or Σ (cf. Chapter 1 in [28] and [30]). Moreover, the class of ω can be read off Δ . Hence, the uncertainty in Δ reflects the non-uniqueness of Kähler classes. Two different constructions are related via Legendre transformations. The Kähler geometry appears naturally when considering the transformation between two different constructions. This was explored by Guillemin ([29]). We will summarize these facts in this section. For simplicity, we only consider toric surfaces, i.e. $n = 2$.

2.1 Toric surfaces and coordinate charts

Let Σ and Δ be a pair consisting of a fan and polytope for a toric surface M . For simplicity, we focus on compact toric surfaces. Then Δ is a Delzant polytope in \mathfrak{t}^* and its closure is compact.

We use the notations in §2.5 ([28]) to describe the fan. Let Σ be a fan given by a sequence of lattice points

$$\{v_0, v_1, \dots, v_{d-1}, v_d = v_0\}$$

in counterclockwise order, in $N = \mathbb{Z}^2 \subset \mathfrak{t}$ such that successive pairs generate N .

Suppose that the vertices and edges of Δ are denoted by

$$\{\vartheta_0, \dots, \vartheta_d = \vartheta_0\}, \quad \{\ell_0, \ell_1, \dots, \ell_{d-1}, \ell_d = \ell_0\}.$$

Here $\vartheta_i = \ell_i \cap \ell_{i+1}$.

By saying that Σ is dual to Δ we mean that v_i is the inward pointing normal vector to ℓ_i of Δ . Hence, Σ is determined by Δ . Suppose that the equation for ℓ_i is

$$l_i(\xi) := \langle \xi, v_i \rangle - \lambda_i = 0. \quad (2.1)$$

Then

$$\Delta = \{\xi | l_i(\xi) > 0, \quad 0 \leq i \leq d-1\}$$

There are three types of cones in Σ : a 0-dimensional cone $\{0\}$ that is dual to Δ ; each of the 1-dimensional cones generated by v_i that is dual to ℓ_i ; each of the 2-dimensional cones generated by $\{v_i, v_{i+1}\}$ that is dual to ϑ_i . We denote them by $Cone_\Delta, Cone_{\ell_i}$ and $Cone_{\vartheta_i}$ respectively.

It is known that for each cone of Σ , one can associate to a complex coordinate chart of M (cf. §1.3 and §1.4 in [28]). Let U_Δ, U_{ℓ_i} and U_{ϑ_i} be the coordinate charts. Then

$$U_\Delta \cong (\mathbb{C}^*)^2; \quad U_{\ell_i} \cong \mathbb{C} \times \mathbb{C}^*; \quad U_{\vartheta_i} \cong \mathbb{C}^2.$$

In particular, in each U_{ℓ_i} there is a divisor $\{0\} \times \mathbb{C}^*$. Its closure is a divisor in M , we denote it by Z_{ℓ_i} .

Remark 2.1 \mathbb{C}^* is called a complex torus and denoted by \mathbb{T}^c . Let z be its natural coordinate. $\mathbb{T} \cong S^1$ is called a real torus.

In this paper, we introduce another complex coordinate by considering the following identification

$$\mathbb{T}^c \rightarrow \mathbb{R} \times 2\sqrt{-1}\mathbb{T}; \quad w = \log z^2. \quad (2.2)$$

We call $w = x + 2\sqrt{-1}y$ the log-affine complex coordinate of \mathbb{C}^* .

When $n = 2$, we have

$$(\mathbb{C}^*)^2 \cong \mathfrak{t} \times 2\sqrt{-1}\mathbb{T}^2.$$

Then (z_1, z_2) on the left hand side is the usual complex coordinate; while (w_1, w_2) on the right hand side is the log-affine coordinate. Write $w_i =$

$x_i + 2\sqrt{-1}y_i$. Then (x_1, x_2) is the coordinate of \mathfrak{t} . In the coordinate (w_1, w_2) , the \mathbb{T}^2 action is given by

$$(t_1, t_2) \cdot (w_1, w_2) = (x_1 + 2\sqrt{-1}(y_1 + t_1), x_2 + 2\sqrt{-1}(y_2 + t_2)),$$

where $(t_1, t_2) \in \mathbb{T}^2$.

We make the following convention.

Remark 2.2 *On different types of coordinate charts, we use different coordinate systems:*

- on $U_\vartheta \cong \mathbb{C}^2$, we use the coordinate (z_1, z_2) ;
- on $U_\ell \cong \mathbb{C} \times \mathbb{C}^*$, we use the coordinate (z_1, w_2) ;
- on $U_\Delta \cong (\mathbb{C}^*)^2$, we use the coordinate (w_1, w_2) , or (z_1, z_2) , where $z_i = e^{\frac{w_i}{2}}$, $i = 1, 2$.

Remark 2.3 *Since we study the \mathbb{T}^2 -invariant geometry on M , it is useful to distinguish a representative point of each \mathbb{T}^2 -orbit. Hence for $(\mathbb{C}^*)^2$, we let the points $\mathfrak{t} \times 2\sqrt{-1}\{1\}$ be the representative points.*

2.2 Kähler geometry on toric surfaces

One can read off the class of the symplectic form of M from Δ . In fact, Guillemin constructed a natural Kähler form ω_o and we denote the class by $[\omega_o]$ and treat it as a canonical point in the class. We call it the Guillemin metric.

For each T^2 -invariant Kähler form $\omega \in [\omega_o]$, on each coordinate chart, there is a potential function (up to linear functions). Let

$$\mathbf{g} = \{g_\Delta, g_{\ell_i}, g_{\vartheta_i} | 0 \leq i \leq d-1\}$$

be the collection of potential functions on U_Δ, U_{ℓ_i} and U_{ϑ_i} for ω_o . For simplicity, we set $g = g_\Delta$.

Let $C_{\mathbb{T}^2}^\infty(M)$ be the set of the smooth \mathbb{T}^2 -invariant functions of M . Set

$$C^\infty(M, \omega_o) = \{\mathbf{f} | \mathbf{f} = \mathbf{g} + \phi, \phi \in C_{\mathbb{T}^2}^\infty(M) \text{ and } \omega_{\mathbf{f}} > 0\}.$$

Here

$$\mathbf{f} = \{(f_\Delta, f_{\ell_i}, f_{\vartheta_i}) | f_\bullet = g_\bullet + \phi\} \text{ and } \omega_{\mathbf{f}} = \omega_{\mathbf{g}} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi.$$

In the definition, \bullet represents any of $\Delta, \ell_i, \vartheta_i$.

Remark 2.4 Let $\phi \in C_{\mathbb{T}^2}^\infty(M)$ and g be any of g_Δ , g_{ℓ_i} , g_{∂_i} . Set $f = g + \phi$. Consider the matrix

$$\mathfrak{M}_f = \left(\sum_k g^{i\bar{k}} f_{j\bar{k}} \right).$$

Though this is not a globally well defined matrix on M , its eigenvalues are globally defined. Set ν_f to be the set of eigenvalues and $H_f = \det \mathfrak{M}_f^{-1}$. These are global functions on M .

Within a coordinate chart with potential function f , the Christoffel symbols, the curvature tensors, the Ricci curvature and the scalar curvature of the Kähler metric ω_f are given by

$$\begin{aligned} \Gamma_{ij}^k &= \sum_{l=1}^n f^{k\bar{l}} \frac{\partial f_{i\bar{l}}}{\partial z_j}, \quad \Gamma_{i\bar{j}}^{\bar{k}} = \sum_{l=1}^n f^{\bar{k}l} \frac{\partial f_{i\bar{l}}}{\partial \bar{z}_j}, \\ R_{i\bar{j}k\bar{l}} &= -\frac{\partial^2 f_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum f^{p\bar{q}} \frac{\partial f_{i\bar{q}}}{\partial z_k} \frac{\partial f_{p\bar{j}}}{\partial \bar{z}_l}, \\ R_{i\bar{j}} &= -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} (\log \det (f_{k\bar{l}})), \quad \mathcal{S} = -\sum f^{i\bar{j}} R_{i\bar{j}}. \end{aligned}$$

respectively. When we use the log-affine coordinates and restrict to \mathfrak{t} ,

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})), \quad \mathcal{S} = -\sum f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})).$$

We treat \mathcal{S} as the operator of scalar curvature of f and denote it by $\mathcal{S}(f)$. Set

$$\mathcal{K} = \|Ric\|_f + \|\nabla Ric\|_f^{\frac{2}{3}} + \|\nabla^2 Ric\|_f^{\frac{1}{2}}, \quad (2.3)$$

$$W = \det(f_{s\bar{t}}), \quad \Psi = \|\nabla \log W\|_f^2. \quad (2.4)$$

On the other hand, we denote by $\dot{\Gamma}_{ij}^k$, $\dot{R}_{k\bar{i}l}^m$ and $\dot{R}_{i\bar{j}}$ the connection, the curvature and the Ricci curvature of the metric ω_o respectively. They have similar formulas as above.

When focusing on U_Δ and using the log-affine coordinate (cf. Remark 2.1), we have $f(x) = g(x) + \phi(x)$. We find that when restricting to $\mathbb{R}^2 \cong \mathbb{R}^2 \times 2\sqrt{-1}\{1\}$, the Riemannian metric induced from ω_f is the Calabi metric G_f (cf. §3.1).

We fix a large constant $K_o > 0$. Set

$$\mathcal{C}^\infty(M, \omega_o; K_o) = \{f \in \mathcal{C}^\infty(M, \omega_o) \mid |\mathcal{S}(f)| \leq K_o\}.$$

2.3 The Legendre transformation, moment maps and potential functions

Let f be a (smooth) strictly convex function on \mathfrak{t} . The gradient of f defines a (normal) map ∇^f from \mathfrak{t} to \mathfrak{t}^* :

$$\xi = (\xi_1, \xi_2) = \nabla^f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right).$$

The function u on \mathfrak{t}^*

$$u(\xi) = x \cdot \xi - f(x).$$

is called the Legendre transform of f . We write $u = L(f)$. Conversely, $f = L(u)$.

The moment map with respect to $\omega_{\mathfrak{f}}$, restricted to U_{Δ} , is given by

$$\tau_{\mathfrak{f}} : U_{\Delta} \xrightarrow{(\log |z_1^2|, \log |z_2^2|)} \mathfrak{t} \xrightarrow{\nabla^f} \Delta \quad (2.5)$$

Note that the first map coincides with (2.2). Let $u = L(f)$. It is known that u must satisfy a certain behavior near the boundary of Δ . In fact, let $v = L(g)$, where g is the potential function of the Guillemin metric; then

Theorem 2.5 (Guillemin) $v(\xi) = \sum_i l_i \log l_i$, where l_i is defined in (2.1).

Then $u = v + \psi$, where $\psi \in C^\infty(\bar{\Delta})$. Set

$$\mathcal{C}^\infty(\Delta, v) = \{u | u = v + \psi \text{ is strictly convex}, \psi \in C^\infty(\bar{\Delta})\}.$$

This space only depends on Δ and we treat v as a canonical point of the space.

We summarize the fact we just presented: let $\mathfrak{f} \in \mathcal{C}^\infty(M, \omega_o)$; then the moment map $\tau_{\mathfrak{f}}$ is given by $f = f_{\Delta}$ via the diagram (2.5) and $u = L(f) \in \mathcal{C}^\infty(\Delta, v)$. In fact, the converse is also true. This is explained in the following.

Given a function $u \in \mathcal{C}^\infty(\Delta, v)$, we can get an $\mathfrak{f} \in \mathcal{C}^\infty(M, \omega_o)$ as the following.

- On U_{Δ} , $f_{\Delta} = L(u)$;
- on U_{ϑ} , f_{ϑ} is constructed in the following steps: (i), suppose that ϑ is the intersection of two edges ℓ_1 and ℓ_2 ; we choose a coordinate system (ξ_1, ξ_2) on \mathfrak{t}^* such that

$$\ell_i = \{\xi | \xi_i = 0\}, i = 1, 2, \quad \Delta \subset \{\xi | \xi_i \geq 0, i = 1, 2\};$$

then u is transformed to be a function in the format

$$u' = \xi_1 \log \xi_1 + \xi_2 \log \xi_2 + \psi';$$

(ii), $f' = L(u')$ defines a function on \mathfrak{t} and therefore is a function on $(\mathbb{C}^*)^2 \subset \mathbb{U}_\vartheta$ in terms of the log-affine coordinate; (iii), it is known that f' can be extended over \mathbb{U}_ϑ and we set f_ϑ to be this function;

- on \mathbb{U}_ℓ , the construction of f_ℓ is similar to f_ϑ . The reader may refer to §2.5 for the construction.

We remark that u and $f = f_\Delta$ are determined by each other. Therefore, all f_θ , f_ℓ can be constructed from f . From this point of view, we may write f for \mathfrak{f} .

2.4 The Abreu equation on Δ

We can transform the scalar curvature operator $\mathcal{S}(f)$ to an operator $\mathcal{S}(u)$ of u on Δ . Then $\mathcal{S}(u)$ is known to be

$$\mathcal{S}(u) = - \sum U^{ij} \partial_{ij}^2 w$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix $(\partial_{ij}^2 u)$, $w = (\det(\partial_{ij}^2 u))^{-1}$. Here and later we denote $\partial_{ij}^2 u = \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}$. The scalar curvature function \mathcal{S} on \mathbb{R}^2 is transformed to a function $\mathcal{S} \circ \nabla^u$ on Δ . Then the equation $\mathcal{S}(f) = \mathcal{S}$ is transformed to be

$$\mathcal{S}(u) = \mathcal{S} \circ \nabla^u.$$

It is well known that $\omega_{\mathfrak{f}}$ gives an extremal metric if and only if $\mathcal{S} \circ \nabla^u$ is a linear function of Δ . Let K be a smooth function on $\bar{\Delta}$; the Abreu equation is

$$\mathcal{S}(u) = K. \tag{2.6}$$

Similarly, we set $\mathcal{C}^\infty(\Delta, v; K_o)$ to be the functions in $u \in \mathcal{C}^\infty(\Delta, v)$ with $|\mathcal{S}(u)| \leq K_o$.

Definition 2.6 *Let K be a smooth function on $\bar{\Delta}$. It is called edge-nonvanishing if it does not vanish on any edge of Δ . That is to say, for any edge ℓ there exists a point $\xi^{(\ell)}$ on the edge such that $K(\xi^{(\ell)}) \neq 0$.*

In our papers, we will always assume that K is edge-nonvanishing.

2.5 A special case: $\mathbb{C} \times \mathbb{C}^*$

Let $\mathfrak{h}^* \subset \mathfrak{t}^*$ be the half plane given by $\xi_1 \geq 0$. The boundary is the ξ_2 -axis and we denote it by \mathfrak{t}_2^* . The corresponding fan consists of only one lattice point $v = (1, 0)$. The coordinate chart is $\mathcal{U}_{\mathfrak{h}^*} = \mathbb{C} \times \mathbb{C}^*$. Let $Z = Z_{\mathfrak{t}_2^*} = \{0\} \times \mathbb{C}^*$ be its divisor.

Let $v_{\mathfrak{h}^*} = \xi_1 \log \xi_1 + \xi_2^2$. Set

$$\mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*}) = \{u | u = v_{\mathfrak{h}^*} + \psi \text{ is strictly convex, } \psi \in C^\infty(\mathfrak{h}^*)\}$$

and $\mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*}; K_o)$ be the functions whose \mathcal{S} is less than K_o .

Take a function $u \in \mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*})$. Then $f = L(u)$ is a function on \mathfrak{t} . Hence it defines a function on the $\mathbb{C}^* \times \mathbb{C}^* \subset \mathcal{U}_{\mathfrak{h}^*}$ in terms of the log-affine coordinate (w_1, w_2) . Then the function $f_{\mathfrak{h}}(z_1, w_2) := f(\log |z_1|^2, \text{Re}(w_2))$ extends smoothly over Z , hence is defined on $\mathcal{U}_{\mathfrak{h}^*}$. We conclude that for any $u \in \mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*})$ it yields a potential function $f_{\mathfrak{h}}$ on $\mathcal{U}_{\mathfrak{h}^*}$.

Using $v_{\mathfrak{h}^*}$ and the above argument, we define a function $g_{\mathfrak{h}}$ on $\mathcal{U}_{\mathfrak{h}^*}$.

3 K-stability and Existence of Extremal Metrics

In a sequence of papers, Donaldson initiates a program to study the extremal metrics on toric manifolds. Here, we sketch his program and some of his important results. Again, we restrict ourself only on the 2-dimensional case.

3.1 K-stability

Let Δ be a Delzant polytope in \mathfrak{t}^* . Most of the material in this subsection can be applied to general convex polytopes, or even convex domains. However, for simplicity we focus on the Delzant polytopes.

For any smooth function K on $\bar{\Delta}$, Donaldson defined a functional on $\mathcal{C}^\infty(\Delta)$:

$$\mathcal{F}_K(u) = - \int_{\Delta} \log \det(\partial_{ij}^2 u) d\mu + \mathcal{L}_K(u),$$

where \mathcal{L}_K is the linear functional

$$\mathcal{L}_K(u) = \int_{\partial\Delta} u d\sigma - \int_{\Delta} K u d\mu,$$

where $d\mu$ is the Lebesgue measure on \mathbb{R}^n and on each face F $d\sigma$ to be a constant multiple of the standard $(n - 1)$ -dimensional Lebesgue measure (see [24] for details). In [24], Donaldson defined the concept of K -stability by using the test configuration (Definition 2.1.2 in [24]). Recall the definition of relatively K -polystability for toric manifolds(cf. [50]).

Definition 3.1 [*relatively K -polystable*] Let $K \in C^\infty(\bar{\Delta})$ be a smooth function on $\bar{\Delta}$. (Δ, K) is called *relatively K -polystable* if $\mathcal{L}_K(u) \geq 0$ for all rational piecewise-linear convex functions u , and $\mathcal{L}_K(u) = 0$ if and only if u is a linear function.

In this paper, we will simply refer to relatively K -polystable as polystable.

We fix a point $p \in \Delta$ and say u is normalized at p if

$$u(p) \geq 0, \quad \nabla u(p) = 0.$$

By Donaldson's work, we make the following definition.

Definition 3.2 (Δ, K) is called *uniformly stable* if it is polystable and for any normalized convex function $u \in \mathcal{C}^\infty(\Delta)$

$$\mathcal{L}_K(u) \geq \lambda \int_{\partial\Delta} u d\sigma$$

for some constant $\lambda > 0$. Sometimes, we say that Δ is (K, λ) -stable.

Donaldson proved

Proposition 3.3 When $n = 2$, if (Δ, K) is polystable and $K > 0$, then there exists a constant $\lambda > 0$ such that Δ is (K, λ) -stable.

This is stated in [24](Proposition 5.2.2).

Conjecture 7.2.2 in [24] reads

Conjecture 3.4 If (Δ, K) is polystable, the Abreu equation $\mathcal{S}(u) = K$ admits a solution in $\mathcal{C}_\infty(\Delta)$, where $\mathcal{C}_\infty(\Delta)$ consists of smooth convex functions on Δ that are continuous on $\bar{\Delta}$.

Note that the difference between $\mathcal{C}_\infty(\Delta)$ and $\mathcal{C}^\infty(\Delta, v)$ is that the second one specifies the boundary behavior of the functions. On the other hand, we proved in [14] the uniform stability is necessary condition. Hence, related to the toric manifolds, we state a stronger version of Conjecture 3.4 for Delzant polytopes.

Conjecture 3.5 *Let Δ be a Delzant polytope. If (Δ, K) is uniformly stable, the Abreu equation $\mathcal{S}(u) = K$ admits a solution in $\mathcal{C}^\infty(\Delta, v)$,*

The conjecture for cscK metric on toric surfaces was recently solved by Donaldson ([25]). In this paper, we solved this conjecture on toric surfaces for any edge-nonvanishing function K .

We need the following result proved by Donaldson.

Theorem 3.6 (Donaldson [27]) *Suppose that Δ is (K, λ) -stable. When $n = 2$, there is a constant $\mathcal{C}_1 > 0$, depending on λ , Δ and $\|\mathcal{S}(u)\|_{C^0}$, such that $\|\max_{\bar{\Delta}} u - \min_{\bar{\Delta}} u\|_{L^\infty} \leq \mathcal{C}_1$.*

3.2 The Continuity Method

We argue the existence of the solution to (2.6) by the standard continuity method.

Let K be the scalar function on $\bar{\Delta}$ in Conjecture 3.5 and suppose that there exists a constant $\lambda > 0$ such that Δ is (K, λ) stable.

Let $I = [0, 1]$ be the unit interval. At $t = 0$ we start with a known metric, for example ω_o (cf. Remark 3.9). Let K_0 be its scalar curvature on Δ . Then Δ must be (K_0, λ_0) stable for some constant $\lambda_0 \geq 0$ (note that we allow that $\lambda_0 = 0$). At $t = 1$, set $(K_1, \lambda_1) = (K, \lambda)$. On Δ , set

$$K_t = tK_1 + (1 - t)K_0, \quad \lambda_t = t\lambda_1 + (1 - t)\lambda_0,$$

it is easy to verify that Δ is (K_t, λ_t) stable. Set

$$\Lambda = \{t | \mathcal{S}(u) = K_t \text{ has a solution in } \mathcal{C}^\infty(\Delta, v).\}$$

Then we should show that Λ is open and closed. Openness is standard by using Lebrun and Simanca's argument ([27, 41]). It remains to get a priori estimates for solutions $u_t = v + \psi_t$ to show closedness. However, for technical reasons, we are only able to prove closedness under the condition that K is an edge-nonvanishing function.

Theorem 3.7 *Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope and (M, ω_o) be the associated compact toric surface. Let $K \in C^\infty(\bar{\Delta})$ be an edge-nonvanishing function and $u_k = v + \psi_k \in \mathcal{C}^\infty(\Delta, v)$ be a sequence of functions with $\mathcal{S}(u_k) = K_k$. Suppose that*

(1) K_k converges to K smoothly on $\bar{\Delta}$;

(2) $\max_{\bar{\Delta}} |u_k| \leq C_1$,

where $C_1 > 0$ is a constant independent of k . Then there is a subsequence of ψ_k which uniformly C^∞ -converges to a function $\psi \in C^\infty(\bar{\Delta})$ with $\mathcal{S}(v + \psi) = K$.

The estimates for ψ_k will be established by the following steps:

Interior estimates: Donaldson proved the interior regularity for the Abreu equation when $n = 2$. In [11], we proved the interior regularity for the Abreu equation of toric manifolds for arbitrary n assuming the C^0 estimates. Due to Theorem 3.6, our result gives an alternative proof of the interior regularity for the $n = 2$ case. The statement is

Theorem 3.8 *Let $u \in C^\infty(\Delta, v)$ and $\mathcal{S}(u) = K$. If $|K|_{C^\infty(\bar{\Delta})} \leq C_o$ and $\max_{\bar{\Delta}} |u| \leq C_1$, then for any $\Omega \subset \subset \Delta$, there is a constant C'_1 , depending on Δ , C_o , C_1 and $d_E(\Omega, \partial\Delta)$, such that*

$$\|u\|_{C^\infty(\Omega)} \leq C'_1.$$

Here, d_E is the Euclidean distance function.

Estimates on edges: This is the most difficult part. On each ℓ , we use the point $\xi^{(\ell)}$ in the interior of ℓ . By the condition, there exists a half ϵ -disk

$$\mathcal{D}^\ell := D_\epsilon(\xi^{(\ell)}) \cap \bar{\Delta}, \quad \overline{\mathcal{D}^\ell} \cap \partial\Delta \subset \ell^o \quad (3.1)$$

such that K is non-zero on this half-disk. Hence there exists a constant δ_o such that

$$|K| > \delta_o, \quad \text{on } \mathcal{D}^\ell. \quad (3.2)$$

Throughout the paper, we will show the regularity on a neighborhood of $\xi^{(\ell)}$ that lies inside \mathcal{D}^ℓ .

Estimates on vertices: Once the first two steps are completed, the regularity on a neighborhood of vertices is based on a subharmonic function. This is done in §9.2.

Remark 3.9 *Let K be an edge-nonvanishing function on Δ . Suppose that it satisfies (3.2). By the computation in Appendix B, we find that we may*

modify ω_o to a new form $\tilde{\omega}_o$ (equivalently, from g to \tilde{g}) such that the scalar curvature \tilde{K}_0 also satisfies (3.2) and

$$\tilde{K}_0 K > 0, \quad \text{on } \mathcal{D}^\ell.$$

Hence we can assume that the whole path K_t connecting \tilde{K}_0 and $K_1 = K$ satisfies (3.2).

Remark 3.10 Since Λ is an open set, there is $t_0 > 0$ such that $[0, t_0) \subset \Lambda$. Obviously $[0, \frac{t_0}{2}] \subset \Lambda$. For any $t \in [\frac{t_0}{2}, 1]$,

$$\lambda_t \geq \min\{\lambda_1, \lambda_{\frac{t_0}{2}}\} > 0, \quad |K_t| \leq |K_0| + |K_1|.$$

Then for any $t \in [\frac{t_0}{2}, 1]$, Δ is (K_t, λ') -stable, where $\lambda' = \min\{\lambda_1, \lambda_{\frac{t_0}{2}}\}$. Let u_t be a solution of the equation $S(u) = K_t$. Applying Theorem 3.6 we have

$$\|\max_{\bar{\Delta}} u_t - \min_{\bar{\Delta}} u_t\|_{L^\infty} \leq \mathcal{C}_1, \quad \forall t \in [\frac{t_0}{2}, 1],$$

where \mathcal{C}_1 is a constant depending only on λ' , Δ and $|K_0| + |K_1|$.

By Theorem 3.7, Remark 3.9 and Remark 3.10 we conclude that Λ is closed. This then implies Theorem 1.1.

We can restate Theorem 3.7 as follows:

Theorem 3.7' Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope and (M, ω_o) be the associated compact toric surface. Let $K \in C^\infty(\bar{\Delta})$ be an edge-nonvanishing function and $\phi_k \in C_{\mathbb{T}^2}^\infty(M)$ be a sequence of functions with

$$\mathcal{S}(f_k) = K_k \circ \nabla^{f_k}, \quad \omega_{f_k} > 0,$$

where $f_k = g + \phi_k$ and g is the potential function of ω_o . Suppose that

(1) K_k converges to K smoothly on $\bar{\Delta}$;

(2) $\max_M |\phi_k| \leq \mathcal{C}_1$,

where $\mathcal{C}_1 > 0$ is a constant independent of k . Then there is a subsequence of ϕ_k which uniformly C^∞ -converges to a function $\phi \in C_{\mathbb{T}^2}^\infty(M)$ with $\mathcal{S}(g + \phi) = K \circ \nabla^f$.

4 Some Results Via Affine techniques

We review the results developed via affine techniques in [9].

4.1 Calabi Geometry

4.1.1. Let $f(x)$ be a smooth, strictly convex function defined on a convex domain $\Omega \subset \mathbb{R}^n \cong \mathfrak{t}$. As f is strictly convex,

$$G := G_f = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$$

defines a Riemannian metric on Ω . We call it the *Calabi* metric. We recall some fundamental facts on the Riemannian manifold (Ω, G) . Let u be the Legendre transform of f and $\Omega^* = \nabla^f(\Omega) \subset \mathfrak{t}^*$. Then it is known that $\nabla^f : (\Omega, G_f) \rightarrow (\Omega^*, G_u)$ is locally isometric. The scalar curvature is $\mathcal{S}(f)$ or $\mathcal{S}(u)$.

Let $\rho = [\det(f_{ij})]^{-\frac{1}{n+2}}$, we introduce the following affine invariants:

$$\Phi = \|\nabla \log \rho\|_G^2 \quad (4.1)$$

$$4n(n-1)J = \sum f^{il} f^{jm} f^{kn} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \frac{\partial^3 f}{\partial x_l \partial x_m \partial x_n} = \sum u^{il} u^{jm} u^{kn} \partial_{ijk}^3 u \partial_{lmn}^3 u. \quad (4.2)$$

Φ is called the norm of the *Tchebychev vector field* and J is called the *Pick invariant*. Put

$$\Theta = J + \Phi. \quad (4.3)$$

4.1.2. Consider an affine transformation

$$\hat{A} : \mathfrak{t}^* \times \mathbb{R} \rightarrow \mathfrak{t}^* \times \mathbb{R}; \quad \hat{A}(\xi, \eta) = (A\xi, \lambda\eta),$$

where A is an affine transformation on \mathfrak{t}^* . If $\lambda = 1$ we call \hat{A} the base-affine transformation. Let $\eta = u(\xi)$ be a function on \mathfrak{t}^* . \hat{A} induces a transformation on u :

$$u^*(\xi) = \lambda u(A^{-1}\xi).$$

Then we have the following lemma of the *affine transformation rule* for the affine invariants.

Lemma 4.1 *Let u^* be as above, then*

1. $\det(\partial_{ij}^2 u^*)(\xi) = \lambda^n |A|^{-2} \det(\partial_{ij}^2 u)(A^{-1}\xi).$
2. $G_{u^*}(\xi) = \lambda G_u(A^{-1}\xi);$
3. $\Theta_{u^*}(\xi) = \lambda^{-1} \Theta_u(A^{-1}\xi);$
4. $\mathcal{S}(u^*)(\xi) = \lambda^{-1} \mathcal{S}(u)(A^{-1}\xi).$

As a corollary,

Lemma 4.2 *G and Θ are invariant with respect to the base-affine transformation. $\Theta \cdot G$ and $\mathcal{S} \cdot G$ are invariant with respect to affine transformations.*

The following lemma was proved in [9].

Lemma 4.3 *Let u be a smooth, strictly convex function defined in \mathbb{R}^n . Suppose that*

$$\partial_{ij}^2 u(0) = \delta_{ij}, \quad \Theta \leq N^2 \quad \text{in } \mathbb{R}^n.$$

Let $\mathbf{a} \geq 1$ be a constant. Let $\lambda_{\min}(\mathbf{a}), \lambda_{\max}(\mathbf{a})$ be the minimal and maximal eigenvalues of $(\partial_{ij}^2 u)$ in $B_{\mathbf{a}}(0)$. Then there exist constants $\mathbf{a}_1, C_1 > 0$, depending only on N and n , such that

- (i) $\exp(-C_1 \mathbf{a}) \leq \lambda_{\min}(\mathbf{a}) \leq \lambda_{\max}(\mathbf{a}) \leq \exp(C_1 \mathbf{a}), \quad \text{in } B_{\mathbf{a}}(0),$
- (ii) $D_{\mathbf{a}_1}(0) \subset B_{\mathbf{a}}(0) \subset D_{\exp(C_1 \mathbf{a})}(0).$

4.1.3. In [9], we used the affine blow-up analysis to prove the following estimates. We only state the results for the Delzant polytopes.

Theorem 4.4 *Let u be a smooth strictly convex function on a Delzant polytope $\Delta \subset \mathbb{R}^2$ with $\|\mathcal{S}(u)\|_{C^3(\Delta)} < K_o$, where $\|\cdot\|_{C^3}$ denotes the Euclidean C^3 -norm. Suppose that for any $p \in \Delta$, $d_u(p, \partial\Delta) < \infty$. Suppose that*

$$\max_{\Delta} u - \min_{\Delta} u \leq C_1 \tag{4.4}$$

for some constant $C_1 > 0$. Then there is a constant $C_3 > 0$, depending only on Δ, C_1, K_o such that

$$(\Theta + |\mathcal{S}| + \mathcal{K})d_u^2(p, \partial\Delta) \leq C_3. \tag{4.5}$$

Here $d_u(p, \partial\Delta)$ is the distance from p to Δ with respect to the Calabi metric G_u .

4.2 Convergence theorems and Bernstein properties

4.2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. It is well-known (see [29], p.27) that there exists a unique ellipsoid E which attains the minimum volume among all the ellipsoids that contain Ω and that is centered at the center of mass of Ω , such that

$$2^{-\frac{3}{2}}E \subset \Omega \subset E,$$

where $2^{-\frac{3}{2}}E$ means the $2^{-\frac{3}{2}}$ -dilation of E with respect to its center. Let T be an affine transformation such that $T(E) = D_1(0)$, the unit disk. Put $\tilde{\Omega} = T(\Omega)$. Then

$$2^{-\frac{3}{2}}D_1(0) \subset \tilde{\Omega} \subset D_1(0). \quad (4.6)$$

We call T the normalizing transformation of Ω .

Definition 4.5 *A convex domain Ω is called normalized when its center of mass is 0 and $2^{-\frac{3}{2}}D_1(0) \subset \Omega \subset D_1(0)$.*

Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an affine transformation given by $A(\xi) = A_0(\xi) + a_0$, where A_0 is a linear transformation and $a_0 \in \mathbb{R}^n$. If there is a constant $L > 0$ such that $|a_0| \leq L$ and for any Euclidean unit vector v

$$L^{-1} \leq |A_0 v| \leq L,$$

we say that A is L -bounded.

Definition 4.6 *A convex domain Ω is called L -normalized if its normalizing transformation is L -bounded.*

The following lemma is useful to measure the normalization of a domain.

Lemma 4.7 *Let $\Omega \subset \mathbb{R}^2$ be a convex domain. Suppose that there exists a pair of constants $R > r > 0$ such that*

$$D_r(0) \subset \Omega \subset D_R(0),$$

then Ω is L -normalized, where L depends only on r and R .

4.2.2. Let u be a convex function on Ω . Let $p \in \Omega$ be a point. Consider the set

$$\{\xi \in \Omega | u(\xi) \leq u(p) + \nabla u(p) \cdot (\xi - p) + \sigma\}.$$

If it is compact in Ω , we call it *a section of u at p with height σ* and denote it by $S_u(p, \sigma)$.

4.2.3. Denote by $\mathcal{F}(\Omega, C)$ the class of convex functions defined on Ω such that

$$\inf_{\Omega} u = 0, \quad u = C \text{ on } \partial\Omega,$$

and

$$\mathcal{F}(\Omega, C; K_o) = \{u \in \mathcal{F}(\Omega, C) | |\mathcal{S}(u)| \leq K_o\}.$$

We will assume that Ω is normalized in this subsection.

The main result of this subsection is the following.

Proposition 4.8 *Let $\Omega \subset \mathbb{R}^2$ be a normalized domain. Let $u \in \mathcal{F}(\Omega, 1; K_o)$ and p^o be its minimal point. Then*

(i) *there are two positive constants s and C_2 such that in $D_s(p^o)$*

$$\|u\|_{C^{3,\alpha}} \leq C_2$$

for any $\alpha \in (0, 1)$; in particular, $d_E(p^o, \partial\Omega) > s$;

(ii) *there is a constant $\delta \in (0, 1)$, such that $S_u(p^o, \delta) \subset D_s(p^o)$.*

(iii) *there exists a constant $b > 0$ such that $S_u(p^o, \delta) \subset B_b(p^o)$.*

In the statement, all the constants only depend on K_o ; D, B are disks with respect to the Euclidean metric and the Calabi metric G_u respectively; d_E is the Euclidean distance function. Here and later we denote $\|\cdot\|_{C^\infty}$ the Euclidean C^∞ -norm, $\|\cdot\|_{C^{3,\alpha}}$ the Euclidean $C^{3,\alpha}$ -norm.

Furthermore, if u is smooth, then

$$\|u\|_{C^\infty(D_s(p^o))} \leq C'_2$$

where C'_2 depends on the C^∞ -norm of $\mathcal{S}(u)$.

This can be restated as a convergence theorem.

Theorem 4.9 *Let $\Omega \subset \mathbb{R}^2$ be a normalized domain. Let $u_k \in \mathcal{F}(\Omega, 1; K_o)$ be a sequence of functions and p_k^o be the minimal point of u_k . Then there exists a subsequence of functions, without loss of generality, still denoted by u_k , converging to a function u_∞ , and p_k^o converging to p_∞^o ; satisfying:*

(i) *there are two positive constants s and C_2 independent of k such that in $D_s(p_\infty^o)$*

$$\|u_k\|_{C^{3,\alpha}} \leq C_2$$

for any $\alpha \in (0, 1)$; in particular, $d_E(p_k^o, \partial\Omega) > s$;

(ii) *there is a constant $\delta \in (0, 1)$, independent of k , such that $S_{u_k}(p_k^o, \delta) \subset D_s(p_\infty^o)$.*

(iii) *there exists a constant $b > 0$ independent of k such that $S_{u_k}(p_k^o, \delta) \subset B_b(p_k^o)$.*

(i) *implies that in $D_s(p_\infty^o)$, u_k C^3 -converges to u_∞ . Furthermore, if u_k is smooth and the C^∞ -norms of $\mathcal{S}(u_k)$ are uniformly bounded, then u_k smoothly converges to u_∞ in $D_s(p_\infty^o)$.*

4.3 Complex interior estimates of W

Let $\Omega \subset \mathbb{C}^n$. Define

$$\mathcal{C}^\infty(\Omega) = \{f \in C^\infty(\Omega) | f \text{ is a real function and } (f_{i\bar{j}}) > 0\}.$$

As a potential function on Ω , f defines a Kähler metric $\omega_f = \frac{\sqrt{-1}}{2} \partial \bar{\partial} f$ on Ω . As explained in §2.2, we have the connections, curvature tensors, etc.

Lemma 4.10 *Let $f \in \mathcal{C}^\infty(\Omega)$ and $B_a(o) \subset \Omega$ be a geodesic ball of radius a centered at o . Suppose that there are constants $N_1, N_2 > 0$ such that*

$$\mathcal{K} \leq N_1, \quad W \leq N_2,$$

in $B_a(o)$. Then in $B_{a/2}(o)$

$$W^{\frac{1}{2}} \Psi \leq C_3 N_2^{\frac{1}{2}} \left[\max_{B_a(o)} \left(|\mathcal{S}| + \|\nabla \mathcal{S}\|_f^{\frac{2}{3}} \right) + a^{-1} + a^{-2} \right].$$

where C_3 is a constant depending only on n and N_1 .

Here \mathcal{K} , W and Ψ are introduced in §2.2. This is proved in [9]. The following lemma is needed at the end of §8.3.

Lemma 4.11 *Let $f \in \mathcal{C}^\infty(\Omega)$ with $\max_{\Omega} |\mathcal{S}(f)| \leq K_o$. Suppose that there is a constant C_0 such that*

$$W \leq C_0, \quad |z| \leq C_0, \quad W(z_0)\mathcal{K} < [100e^{C_0^2}]^{-1} \quad (4.7)$$

in $B_1(z_0)$. Then there is a constant $C_1 > 0$ depending only on n, K_o and C_0 , such that

$$W(z_0) \geq C_1. \quad (4.8)$$

Proof. Consider the function

$$F = (a^2 - r^2)^2 e^{\sum z_i \bar{z}_i} W^{-\frac{1}{2}},$$

defined in $B_a(z_0)$, where $r(z)$ is the geodesic distance from z_0 to z . F attains its supremum at some interior point p^* . We have, at p^* ,

$$\begin{aligned} -\frac{1}{2}(\log W)_{,k} + \left(\sum z_i \bar{z}_i\right)_{,k} - \frac{4(r^2)_{,k}}{a^2 - r^2} &= 0, \\ \frac{\mathcal{S}}{2} + \sum f^{i\bar{i}} - \frac{4\|(r^2)_{,k}\|_f^2}{(a^2 - r^2)^2} - \frac{4\Box(r^2)}{a^2 - r^2} &\leq 0. \end{aligned} \quad (4.9)$$

where \Box is the Laplacian operator with respect to the metric ω_f . Note that $\sum f^{i\bar{i}} \geq 2W^{-\frac{1}{2}}$. We apply the Laplacian comparison Theorem for $r\Box r$: since the norm of the Ricci curvature $\text{Ric} \geq -\epsilon W^{-1}(z_0)$, where $\epsilon = [100e^{C_0^2}]^{-1}$.

$$r\Box r \leq (1 + \epsilon^{\frac{1}{2}} W^{-\frac{1}{2}}(z_0)r).$$

Using the facts $|S| \leq K_o$ and $\|\nabla r\|_f = 1$, we have

$$W^{-\frac{1}{2}} \leq C_2 + \frac{C'_2(n)a^2}{(a^2 - r^2)^2} + \frac{8a\epsilon^{\frac{1}{2}}W^{-\frac{1}{2}}(z_0)}{a^2 - r^2}. \quad (4.10)$$

Note that F attains its maximum at p^* . Then $F(z_0) \leq F(p^*)$. Let $a = 1$ and if ϵ is small, the lemma follows from a direct calculation. ■

4.4 Interior regularities and estimate of \mathcal{K} near divisors

Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope. In [9], we prove the following regularity theorem.

Theorem 4.12 *Let \mathcal{U} be a chart of either $\mathcal{U}_\Delta, \mathcal{U}_\ell$ or \mathcal{U}_ϑ . Let $z_o \in \mathcal{U}$ and $B_a(z_o)$ be a geodesic ball in \mathcal{U} . Suppose that there is a constant C_1 such that $f(z_o) = 0$, $|\nabla f|(z_o) \leq C_1$, and*

$$\mathcal{K}(f) \leq C_1, \quad C_1^{-1} \leq W \leq C_1, \quad |z| \leq C_1$$

in $B_a(z_o)$. Then there is a constant $a_1 > 0$, depending on a and C_1 , such that

$$\begin{aligned} \|f\|_{C^{3,\alpha}(D_{a_1}(z_o))} &\leq C(a, C_1, \|\mathcal{S}(f)\|_{C^0}), \\ \|f\|_{C^\infty(D_{a_1}(z_o))} &\leq C(a, C_1, \|\mathcal{S}(u)\|_{C^\infty}). \end{aligned}$$

One of the main results in [9] that is developed from the affine technique is the following.

Theorem 4.13 *Let $u \in C^\infty(\Delta, v)$. Let \mathfrak{z}_* be a point on a divisor Z_ℓ for some ℓ . Choose a coordinate system (ξ_1, ξ_2) such that $\ell = \{\xi | \xi_1 = 0\}$. Let $p \in \ell$ and $D_b(p) \cap \bar{\Delta}$ be an Euclidean half-ball such that its intersects with $\partial\Delta$ lies in the interior of ℓ . Let $B_a(\mathfrak{z}_*)$ be a geodesic ball satisfying $\tau_f(B_a(\mathfrak{z}_*)) \subset D_b(p)$. Suppose that*

$$\begin{aligned} |\mathcal{S}(u)| &\geq \delta > 0, \text{ in } D_b(p) \cap \bar{\Delta}, \\ \|\mathcal{S}(u)\|_{C^3(\bar{\Delta})} &\leq \mathbf{N}, \\ \partial_{22}^2 h|_{\ell \cap D_b(p)} &\geq \mathbf{N}^{-1} \end{aligned} \tag{4.11}$$

where $h = u|_\ell$ and $\|\cdot\|_{C^3(\Delta)}$ denotes the Euclidean C^3 -norm. Then there is a constant $\mathbf{C}_3 > 0$, depending only on a, δ, \mathbf{N} and $D_b(p)$, such that

$$\frac{\min_{B_a(\mathfrak{z}_*) \cap Z_\ell} W}{\max_{B_a(\mathfrak{z}_*)} W} (\mathcal{K}(\mathfrak{z}) + \|\nabla \log |\mathcal{S}|\|_f^2(\mathfrak{z})) a^2 \leq \mathbf{C}_3, \quad \forall \mathfrak{z} \in B_{a/2}(\mathfrak{z}_*) \tag{4.12}$$

where $W = \det(f_{s\bar{t}})$.

5 Estimates of the Determinant

As stated in Theorem 3.8, the interior regularity in Δ has been established. In this section, we will explore the dependence of some estimates on $d_E(\cdot, \partial\Delta)$. The results in §5.1 hold for any n .

5.1 The Lower Bound of the Determinant

The following lemma can be found in [26].

Lemma 5.1 *Suppose that $u \in \mathcal{C}^\infty(\Delta, v; K_o)$. Then*

- (1) $\det(\partial_{ij}^2 u) \geq C_4$ everywhere in Δ , where $C_4 = (4n^{-1}K_o \text{diam}(\Delta)^2)^{-n}$.
- (2) For any $\delta \in (0, 1)$ there is a constant $C_\delta > 0$, depending only on n and δ , such that

$$\det(\partial_{ij}^2 u)(p) \geq C_\delta d_E(p, \partial\Delta)^{-\delta}.$$

Here we denote $\partial_{ij}^2 u = \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}$. Let $q \in \partial\Delta$ be the point such that $d_E(p, q) = d_E(p, \partial\Delta)$. When q is an interior point of a face E , we derive a stronger estimate for $\det(\partial_{ij}^2 u)(p)$.

Lemma 5.2 *Let $u \in \mathcal{C}^\infty(\Delta, v; K_o)$. Let F be a facet of Δ and q be a point in its interior. Let ϵ be a small constant such that $D_\epsilon(q) \cap \bar{\Delta}$ is a half-ball and $D_\epsilon(q) \cap \partial\Delta \subset F$. Then there is a constant $C_5 > 0$, depending only on Δ, K_o, ϵ , such that for any ξ in this half-disk*

$$\det(\partial_{ij}^2 u)(\xi) \geq \frac{C_5}{d_E(\xi, F)}.$$

Proof. Without loss of generality, we choose a new coordinate system on \mathbf{t}^* such that: (i) $\xi(q) = 0$; (ii) F is on the $\{\xi_1 = 0\}$ -plane; (iii) $\xi_1(\Delta) \geq 0$.

By (2) of Lemma 5.1, we already have

$$\det(\partial_{ij}^2 u) \geq C_0 \xi_1^{-(1-\frac{1}{n})}. \quad (5.1)$$

Consider the function

$$v' = \xi_1^\alpha \left(C + \sum_{j=2}^n \xi_j^2 \right) - a\xi_1,$$

where $a > 0$, $\alpha > 1$ and $C > 0$ are constants to be determined. We may choose a large such that $v' \leq 0$ on Δ . For any point ξ we may assume that $\xi = (\xi_1, \xi_2, 0, \dots, 0)$. By a direct calculation we have

$$v'_{11} = \alpha(\alpha - 1)\xi_1^{\alpha-2}(C + \xi_2^2), \quad v'_{12} = 2\alpha\xi_2\xi_1^{\alpha-1}, \quad v'_{ii} = 2\xi_1^\alpha \quad i \geq 2,$$

$$\det(\partial_{ij}^2 v') = 2^{n-1} [\alpha(\alpha - 1)(C + \xi_2^2) - 2\alpha^2\xi_2^2] \xi_1^{n\alpha-2}.$$

Set $\alpha = 1 + \frac{1}{n^2}$. Then for large C , it is easy to see that v' is strictly convex in Δ and

$$\det(\partial_{ij}^2 v') \geq C_1 \xi_1^{n\alpha-2}. \quad (5.2)$$

Consider the function

$$F = w + C_5 v',$$

where $w^{-1} = \det(\partial_{ij}^2 u)$. As w vanishes on the boundary of Δ , we have $F \leq 0$ on $\partial\Delta$. Then

$$\begin{aligned} \sum U^{ij} \partial_{ij}^2 F &= -K + C_5 \det(\partial_{ij}^2 u) \sum u^{ij} \partial_{ij}^2 v' \\ &\geq -K + nC_5 \det(\partial_{ij}^2 u)^{1-1/n} \det(\partial_{ij}^2 v')^{1/n} \\ &\geq -K + nC_5 C' \xi_1^{-(1-\frac{1}{n})^2} C_1 \xi_1^{\alpha-\frac{2}{n}} \\ &= -K + nC_5 C' C_1. \end{aligned}$$

Here (u^{ij}) denotes the inverse matrix of the matrix $(\partial_{ij}^2 u)$. Choose C_5 such that $\sum U^{ij} \partial_{ij}^2 F > 0$. So by the maximum principle we have $w \leq C_5 |v'| \leq aC_5 \xi_1$. It follows that $\det(\partial_{ij}^2 u) \geq C_5 \xi_1^{-1}$ for some constant $C_5 > 0$. ■

5.2 The Upper Bound of the Determinant

Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope. We need the following two lemmas.

Lemma 5.3 *Let $\Omega \subset \mathbb{R}^n$ and $p \in \Omega^\circ$ be a point with $\xi(p) = 0$. Let $u_k \in C^\infty(\Omega)$ be a sequence of functions such that*

$$u_k(0) = 0, \quad \nabla u_k(0) = 0,$$

and u_k locally uniformly C^2 -converges to a strictly convex function u_∞ in Δ° . Then there are two constants $d, C_1 > 0$ independent of k such that

$$\frac{\sum \left(\frac{\partial u_k}{\partial \xi_i} \right)^2}{(d + f_k)^2} \leq C_1$$

where f_k is the Legendre function of u_k (cf. Section 2.3).

Proof. Obviously $f_k(0) = 0$, $\nabla f_k(0) = 0$, $f_k \geq 0$ and f_k uniformly C^2 -converges to a strictly convex function f_∞ in $D_\epsilon(0)$ for some $\epsilon > 0$, in particular,

$$f_k|_{\partial D_\epsilon(0)} \geq \delta$$

for some $\delta > 0$ independent of k . Let $h(x) = \frac{\delta|x|}{\epsilon}$. Using the convexity of f_k one can check that in $\mathbb{R}^n \setminus D_\epsilon(0)$,

$$f_k(x) \geq h(x).$$

Then in \mathbb{R}^n

$$(f_k + \delta) \frac{\epsilon}{\delta} \geq |x| \geq \frac{(\sum x_i^2)^{\frac{1}{2}}}{n}. \quad q.e.d.$$

The following lemma is proved in [10]:

Lemma 5.4 *Suppose that $u \in C^\infty(\Delta, v; K_o)$, and suppose that there are two constants $b, d > 0$ such that*

$$\frac{\sum \left(\frac{\partial u}{\partial \xi_k} \right)^2}{(d + f)^2} \leq b, \quad d + f \geq 1 \quad (5.3)$$

where f is the Legendre function of u . Then there is a constant $b_0 > 0$ depending only on K_o and Δ such that

$$\frac{\det(\partial_{ij}^2 u)}{(d + f)^4}(p) \leq \frac{b_0}{d_E(p, \partial\Delta)^4}.$$

Using Lemma 5.3 and Lemma 5.4 we can obtain the upper bound estimates for $\det(\partial_{ij}^2 u)$.

Lemma 5.5 *Suppose that $u_k \in C^\infty(\Delta, v; K_o)$ and u_k locally uniformly C^2 -converges to a strictly convex function u_∞ in Δ° . And suppose that*

$$\max_{\Delta} |u_k| \leq \mathcal{C}_1.$$

for some constant $\mathcal{C}_1 > 0$ independent of k . Denote $d_E(p, \partial\Delta)$ by the Euclidean distance from p to the boundary $\partial\Delta$. Then there is a constant $\mathcal{C}_6 > 0$, independent of k , such that for any $p \in \Delta^\circ$

$$\log \det(\partial_{ij}^2 u_k)(p) \leq \mathcal{C}_6 - \mathcal{C}_6 \log d_E(p, \partial\Delta).$$

Proof. Since u is convex, we have for any $p \in \Delta^\circ$

$$\left| \frac{\partial u}{\partial \xi_i}(p) \right| \leq \frac{2\mathcal{C}_1}{d_E(p, \partial\Delta)}. \quad (5.4)$$

Again by the convexity and (5.4) we obtain that for any point $p \in \Delta^\circ$

$$f(\nabla_u(p)) = \sum \frac{\partial u}{\partial \xi_i} \xi_i - u \leq C_2(1 + |x_1| + |x_2|) \leq \frac{4C_2\mathcal{C}_1}{d_E(p, \partial\Delta)}, \quad (5.5)$$

where $C_2 > 0$ is a constant depending only on Δ . From Lemma 5.3, Lemma 5.4 and (5.5) we conclude that

$$\det(\partial_{ij}^2 u) \leq C_3(d_E(p, \partial\Delta))^{-8}, \quad (5.6)$$

where $C_3 > 0$ is a constant. q.e.d.

6 Estimates of Riemannian distances on $\partial\Delta$

Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope. Let ℓ be an edge of Δ . As explained in §3.2, we assume that there is a point $\xi^{(\ell)}$ and a neighborhood \mathcal{D}^ℓ . We establish some important facts in \mathcal{D}^ℓ . For simplicity, we fix a coordinate system on \mathfrak{t}^* such that (i) ℓ is on the ξ_2 -axis; (ii) $\xi^{(\ell)} = 0$; (iii) $\Delta \subset \mathfrak{h}^*$.

Define $\ell_{c,d} = \{(0, \xi_2) | c \leq \xi_2 \leq d\} \subset \ell^\circ$.

Let $u_k \in \mathcal{C}^\infty(\Delta, v; K_o)$ be a sequence of functions with $\mathcal{S}(u_k) = K_k$. Suppose that

$$\left| \max_{\Delta} u_k - \min_{\Delta} u_k \right| \leq \mathcal{C}_1 \quad (6.1)$$

for some \mathcal{C}_1 constant independent of k , K_k uniformly C^3 -converges to K on $\bar{\Delta}$ and u_k locally uniformly C^6 -converges in Δ to a strictly convex function u_∞ . u_∞ can be naturally continuously extended to be defined on $\bar{\Delta}$.

6.1 C^0 Convergence

Denote by h_k the restriction of u_k to ℓ . Then h_k locally uniformly converges to a convex function h in ℓ° . Obviously, $u_\infty|_{\ell^\circ} \leq h$. In this subsection we prove that " \leq " is indeed " $=$ ". In fact, we have the following Proposition.

Proposition 6.1 *For $q \in \ell^\circ$, $u_\infty(q) = h(q)$.*

Proof. For simplicity denote

$$\xi(q) = 0, \quad \mathcal{S}(u_k) = K_k. \quad (6.2)$$

If this proposition is not true, then $u_\infty(0) < h(0)$. Without loss of generality we can assume that $\ell_{-\frac{1}{2}, \frac{1}{2}} = \{\xi | \xi_1 = 0, |\xi_2| \leq \frac{1}{2}\} \subset \ell^o$ and for any point $p \in \ell_{-\frac{1}{2}, \frac{1}{2}}$

$$u_\infty(p) + \frac{1}{2} < h(p), \quad u_\infty(0) = 0.$$

By assumption we have

$$\lim_{k \rightarrow \infty} \|K_k - K\|_{C^0(\bar{\Delta})} = 0. \quad (6.3)$$

For any K_k , consider the functional

$$\mathcal{F}_k(u) = \mathcal{F}_{K_k}(u) = - \int_{\Delta} \log \det(\partial_{ij}^2 u) d\mu + \mathcal{L}_{K_k}(u),$$

defined in $C^\infty(\Delta)$, where \mathcal{L}_k is the linear functional

$$\mathcal{L}_{K_k}(u) = \int_{\partial\Delta} u d\sigma - \int_{\Delta} K_k u d\mu.$$

Here $d\sigma$ and $d\mu$ are as in the section 3.1. Since u_k satisfies (6.2), by a result of Donaldson u_k is an absolute minimiser for \mathcal{F}_k (cf. [24]). By (6.1) and $u_\infty(0) = 0$

$$|u_k|_{L^\infty(\bar{\Delta})} < \mathcal{C}_1 + 1 \quad (6.4)$$

as k large enough. For any positive constant $\delta < 1$, denote

$$\Delta_\delta = \{p \in \Delta \mid d_E(p, \partial\Delta) \geq \delta\}.$$

By Lemma 5.1 and Lemma 5.5 we have for any k

$$\left| \int_{\Delta \setminus \Delta_\delta} \log \det(\partial_{ij}^2 u_k) d\mu \right| \leq C_2 \sqrt{\delta}$$

as δ is small enough, where C_2 is a constant independent of k and δ . Combing this and that u_k locally uniformly C^3 -converges to u_∞ in Δ we have

$$\lim_{k \rightarrow \infty} \int_{\Delta} \log \det(\partial_{ij}^2 u_k) d\mu = \int_{\Delta} \log \det(\partial_{ij}^2 u_\infty) d\mu.$$

By (6.3) and (6.4) we have

$$\lim_{k \rightarrow \infty} \int_{\Delta} S_k u_k d\mu = \int_{\Delta} S_{\infty} u_{\infty} d\mu$$

and

$$\lim_{k \rightarrow \infty} \int_{\partial \Delta} u_k d\sigma - \int_{\partial \Delta} u_{\infty} d\sigma = \int_{\partial \Delta} h d\sigma - \int_{\partial \Delta} u_{\infty} d\sigma \geq \int_{\ell_{-\frac{1}{2}, \frac{1}{2}}} (h - u_{\infty}) d\sigma \geq \frac{1}{2}.$$

We conclude that

$$\mathcal{F}_{S_{\infty}}(u_{\infty}) \leq \lim_{k \rightarrow \infty} \mathcal{F}_k(u_k) - \frac{1}{2}. \quad (6.5)$$

Hence

$$\mathcal{F}_k(u_{\infty}) = \mathcal{F}_{S_{\infty}}(u_{\infty}) - \int_{\Delta} (\mathcal{S}_k - \mathcal{S}_{\infty}) u_{\infty} d\mu \leq \mathcal{F}_k(u_k) - \frac{1}{4}, \quad (6.6)$$

as k large enough, where we used (6.3) and (6.4) in the last inequality. This contradicts $\mathcal{F}_k(u_k) = \inf_{u \in C^{\infty}(\Delta)} \mathcal{F}(u)$. ■

6.2 Monge-Ampère measure on the boundary

Lemma 6.2 *Let $u \in \mathcal{C}^{\infty}(\Delta, v; K_o)$ and $h = u|_{\ell}$. There is a constant $C_7 > 0$, depending on $d_E(\ell_{c,d}, \partial \ell)$, such that on $\ell_{c,d}$, $\partial_{22}^2 h \geq C_7$.*

Proof. By the boundary behavior of u , we know that $\frac{\partial u^{11}}{\partial \xi_1} = 1$ on ℓ (cf. [27]). Consider a small neighborhood of $\ell_{c,d}$ such that $\frac{1}{2} \leq \frac{\partial u^{11}}{\partial \xi_1} \leq 2$. By integrating we have, in this neighborhood,

$$\frac{1}{2} \xi_1 \leq u^{11} \leq 2 \xi_1.$$

Then, by Lemma 5.2, $\partial_{22}^2 u = \det(\partial_{ij}^2 u) \cdot u^{11} \geq C_7$. ■

6.3 A lemma

Let $u \in \mathcal{C}^{\infty}(\mathbf{h}^*, v_{\mathbf{h}^*}; K_o)$. Let p° be a point such that $d(p^{\circ}, \mathbf{t}_2^*) = 1$, where $\mathbf{t}_2^* = \partial \mathbf{h}^*$. By adding a linear function we normalize u such that p° is the minimal point of u ; i.e.,

$$u(p^{\circ}) = \inf_{\mathbf{h}^*} u. \quad (6.7)$$

Let \check{p} be the minimal point of u on \mathfrak{t}_2^* , the boundary of \mathfrak{h}^* . By adding some constant to u , we require that

$$u(\check{p}) = 0. \quad (6.8)$$

By a coordinate translation we can assume that

$$\xi(\check{p}) = 0. \quad (6.9)$$

We call (u, p°, \check{p}) a normalized triple, if u satisfies (6.7), (6.8), (6.9) and

$$d(p^\circ, \mathfrak{t}_2^*) = 1.$$

By the same methods as in §7.3 ([9]), we can prove that: for any sequence $(u_k, p_k^\circ, \check{p}_k)$ of normalized triples with

$$\lim_{k \rightarrow \infty} \max |\mathcal{S}(u_k)| = 0, \quad \Theta(p) d^2(p, \mathfrak{t}_2^*) \leq \mathbf{C}_3, \quad (6.10)$$

Then there is a constant $C > 1$ such that

$$C^{-1} \leq |u_k(p_{ok})| \leq C. \quad (6.11)$$

Based on this, we prove the following lemma in this subsection.

Lemma 6.3 *Let $(u_k, p_k^\circ, \check{p}_k)$ be a sequence of normalized triple with (6.10) and $\partial_{ij}^2 u_k(p_k^\circ) = \delta_{ij}$. Then by choosing a subsequence we have*

- (1) p_k° converges to a point p_∞° , and u_k locally uniformly C^3 -converges to a strictly convex function u_∞ , in particular, u_k uniformly C^3 -converges to u_∞ in a Euclidean ball $D_a(p_\infty)$ for some $a > 0$;
- (2) there exist two constants $0 < \tau < 1$ and $C_1 > 0$ independent of k such that

$$\max_{S_h(\check{p}, 1)} |\xi_2| \leq \frac{C_1}{2}, \quad (6.12)$$

$$\max_{S_h(\check{p}, 1)} \partial_2 u \geq C_1^{-1}, \quad \min_{S_h(\check{p}, 1)} \partial_2 u \leq -C_1^{-1}, \quad (6.13)$$

$$|\nabla u| \leq C_1^{-1}, \quad \text{in } B_\tau(p^\circ) \quad (6.14)$$

as k large enough, where $h = u|_{\mathfrak{t}_2^*}$ and $S_h(\check{p}, 1) = \{\xi \in \mathfrak{t}_2^* \mid h \leq 1\}$.

Proof. Let u be a function of a sequence functions u_k . By a coordinate translation $A(\xi_1, \xi_2) = (\xi_1 - \alpha, \xi_2 - \beta)$, we can assume that $\xi(p^\circ) = 0$.

By (6.10) and $d(p^\circ, \mathfrak{t}_2^*) = 1$, we have

$$\Theta \leq 16C_3, \text{ in } B_{\frac{3}{4}}(p^\circ).$$

Using Lemma 4.3, (6.11) and a direct integration we conclude that

$$\|u\|_{C^{3,\alpha}(B_{\frac{1}{2}}(p^\circ))} \leq C_2$$

for some positive constant C_2 independent of k . Then by Ascoli Theorem and choosing a subsequence we conclude that u_k uniformly C^3 -converges to a strictly convex function u_∞ in $D_a(0)$. In particular, there is a positive constant ϵ such that

$$S_{u_k}(0, \epsilon) \subset D_a(0), \quad \left| \frac{\partial u_k}{\partial e_r} \right| (p) \geq \frac{\epsilon}{2a}, \quad \forall p \in \partial D_a(0). \quad (6.15)$$

where $e_r = \frac{\xi}{|\xi|}$. Consider the function

$$\Lambda(\xi) = \frac{\epsilon}{2a}|\xi| - C,$$

where C is the constant in (6.11). Since $\Lambda(0) < u(0)$ and $\Lambda(p) \leq u(p)$ for any $p \in \partial D_a(0)$, by (6.15) and the convexity of u we have

$$\Lambda(q) \leq u(q), \quad \forall q \in \mathfrak{h}^* \setminus D_a(0). \quad (6.16)$$

Then by $\mathfrak{t}_2^* \subset \mathfrak{h}^* \setminus D_a(0)$, we have $S_h(\check{p}, 1) \subset S_\Lambda(p^\circ, C+1) \cap \mathfrak{t}_2^*$. In particular

$$\max_{S_h(\check{p}, 1)} |\xi| \leq \max_{S_\Lambda(p^\circ, C+1)} |\xi| \leq \frac{2a}{\epsilon}(C+1). \quad (6.17)$$

Combing this and $(-\alpha, -\beta) \in S_h(\check{p}, 1)$, we prove (6.12). (1) follows from (6.17) and the convergence of u_k . (6.14) follows from (1) and the convexity of u_∞ . Then by the convexity of u and (6.12) we obtain (6.13). \square

6.4 Lower Bounds of Riemannian Distances inside edges

Let $p = (0, c)$ and $q = (0, d)$. Let ϵ_o be a constant such that $\ell_{c,d} \subset \ell_{c-2\epsilon_o, d+2\epsilon_o} \subset \subset \ell$ and $\epsilon_o \leq \frac{d-c}{4}$. Set

$$E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_o} = [0, \delta_o] \times \ell_{c-2\epsilon_o, d+2\epsilon_o} \subset \overline{\Delta}.$$

We assume that $E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_o} \cap (\partial\Delta \setminus \ell) = \emptyset$.

We use affine technique to prove the following proposition.

Proposition 6.4 *There is a constant $C_8 > 0$ independent of k such that*

$$d_{u_k}(p, q) \geq C_8$$

for k large enough. Here $d_{u_k}(p, q)$ denotes the geodesic distance from p to q with respect the Calabi metric G_{u_k} .

We introduce some notations. Let u be a function of the sequence u_k . Let Γ be a minimal geodesic from $(0, c)$ to $(0, d)$ with respect to the Calabi metric G_u . For any $p^\circ \in \Gamma \setminus \ell$, denote

$$d(p^\circ) = d(p^\circ, \partial\Delta).$$

Let $\check{p} \in \ell$ be the point such that

$$\partial_2 u(\check{p}) = \partial_2 u(p^\circ). \quad (6.18)$$

Let $L(p^\circ)$ be the geodesic arc-length of the connected component containing p° of $\Gamma \cap B_{\tau d(p^\circ)}(p^\circ)$, where τ is the constant in Lemma 6.3. Then $L(p^\circ) = 2\tau d(p^\circ)$. Denote

$$m(p^\circ) = \max_{q \in B_{\tau d(p^\circ)}(p^\circ)} |\partial_2 u(q) - \partial_2 u(p^\circ)|,$$

$$\mathcal{B}(\check{p}) = \{p \in \ell \mid |\partial_2 u(p) - \partial_2 u(\check{p})| < m(p^\circ)\}.$$

To prove Proposition 6.4 we need the following lemma.

Lemma 6.5 *Let $u_k \in C^\infty(\Delta, v; K_o)$ be a sequence of functions such that u_k satisfies (6.1) and*

$$\mathcal{S}(u_k) = K_k.$$

Suppose that K_k uniformly C^3 -converges to K on $\overline{\Delta}$ and the geodesic arc-length of Γ_k converges to zero as $k \rightarrow \infty$. Then there is a positive constant C_2 independent of k such that for any u_k and any $p^\circ \in \Gamma_k \cap E_{c-\epsilon_o, d+\epsilon_o}^{\delta_o}$,

$$\int_{\mathcal{B}_k(\check{p})} \sqrt{\partial_{22}^2 u_k} d\xi_2 \leq C_2 \tau d_k(p^\circ) \leq C_2 L_k(p^\circ), \quad (6.19)$$

where $d_k(p^\circ) = d_{u_k}(p^\circ, \partial\Delta)$ and \check{p} satisfies (6.18).

Proof. If the lemma is not true, there are a subsequence of points p_k° and a subsequence of functions u_k , still denoted by p_k° and u_k to simplify notations, such that

$$\lim_{k \rightarrow \infty} \frac{\tau d_k(p_k^\circ)}{\int_{\mathcal{B}(\check{p}_k)} \sqrt{\partial_{22}^2 u_k} d\xi_2} = 0. \quad (6.20)$$

Let u be a function of the sequence u_k . Let $\hat{u} = u - \nabla u(p^\circ) \cdot \xi + C$, where C is a constant such that $\inf_{\partial\Delta} \hat{u} = 0$. Then, $\hat{u}(\check{p}) = \inf_\ell \hat{u}$, $\hat{u}(p^\circ) = \inf_\Delta \hat{u}$. We claim that

$$\inf_{\partial\Delta} \hat{u} = \inf_\ell \hat{u} = \hat{u}(\check{p}), \quad (6.21)$$

as k is large enough.

Proof of the Claim. If the Claim is not true, there are a subsequence of \hat{u}_k and a sequence of points $\check{q}_k \in \partial\Delta \setminus \ell$, still denoted by \hat{u}_k and \check{q}_k , such that

$$\hat{u}_k(\check{q}_k) = \inf_{\partial\Delta} \hat{u}_k.$$

Let $\alpha_k = -\inf_\Delta \hat{u}_k$. Since the geodesic arc-length of Γ_k converges to zero as $k \rightarrow \infty$ we have $\lim_{k \rightarrow \infty} d_k(p_k^\circ) = 0$. Then by the interior regularity,

$$\lim_{k \rightarrow \infty} p_k^\circ = \check{p}_\infty \in \ell_{c-\epsilon_o, d+\epsilon_o}. \quad (6.22)$$

It follows from (6.22) and Proposition 6.1 that

$$\lim_{k \rightarrow \infty} |u_k(p_k^\circ) - u_k(\check{p}_\infty)| = 0. \quad (6.23)$$

By the convexity of u and (6.1) we have $|\partial_2 u(p^\circ)| \leq C_1$ for some positive constant C_1 independent of k . Then

$$\begin{aligned} & \hat{u}(\check{p}_\infty) - \hat{u}(p^\circ) \\ &= u(\check{p}_\infty) - u(p^\circ) + \partial_1 u(p^\circ) \xi_1(p^\circ) - \partial_2 u(p^\circ) (\xi_2(\check{p}_\infty) - \xi_2(p^\circ)) \\ &\leq u(\check{p}_\infty) - u(p^\circ) + C_1 |(\xi_2(\check{p}_\infty) - \xi_2(p^\circ))| \end{aligned} \quad (6.24)$$

as k is large enough, where we used the fact $\partial_1 u(p^\circ) < 0$, $\xi_1(p^\circ) > 0$. Combining (6.22), (6.23), (6.24) and $\hat{u}(\check{p}_\infty) > \hat{u}(\check{q}) > \hat{u}(p^\circ)$ we conclude that

$$\lim_{k \rightarrow \infty} \alpha_k = 0. \quad (6.25)$$

Let γ be the line segment connecting p° and \check{q} . By the convexity we have for any $p \in \gamma$

$$-\alpha \leq \hat{u}(p) \leq 0, \quad l(p) \leq u(p) \leq l(p) + \alpha, \quad (6.26)$$

where $l(p) = u(p^\circ) + \nabla u(p^\circ) \cdot (p - p^\circ)$. By choosing a subsequence we can assume that γ_k converges to a line segment γ_∞ . We can see that $\gamma_\infty \subset \ell$, otherwise, as $\lim_{k \rightarrow \infty} \alpha_k = 0$, u_∞ is a linear function on γ_∞ , it contradicts to the strictly convexity of u_∞ . For any $p \in \gamma_\infty \cap \ell_{c-\epsilon_o, d+\epsilon_o}$ and $p_k \in \gamma_k$ with $\lim_{k \rightarrow \infty} p_k = p$, by the same calculation as (6.24) we have

$$0 \leq \lim_{k \rightarrow \infty} (\hat{u}_k(p) - \hat{u}_k(p_k)) \leq \lim_{k \rightarrow \infty} (u_k(p) - u_k(p_k)) = 0. \quad (6.27)$$

Let $\hat{h} = \hat{u}|_\ell$. By (6.25), (6.26) and (6.27) we conclude that

$$\lim_{k \rightarrow \infty} \hat{h}_k(p) = 0, \quad \forall p \in \ell_{c-\epsilon_o, d+\epsilon_o}. \quad (6.28)$$

On the other hand, since $\hat{h} = h - \partial_2 u(p^\circ)(\xi_2 - \xi_2(p^\circ)) + C$, \hat{h}_k converges to a strictly convex function \hat{h}_∞ . It contradicts to (6.28). The Claim is proved.

By a coordinate translation, we assume that $\xi(\check{p}) = 0$. Consider the following affine transformation T ,

$$\tilde{\xi}_1 = a_{11}\xi_1, \quad \tilde{\xi}_2 = a_{21}\xi_1 + a_{22}\xi_2, \quad \tilde{u}(\tilde{\xi}) = \lambda \hat{u} \left(\frac{\tilde{\xi}_1}{a_{11}}, \frac{\tilde{\xi}_2}{a_{22}} - \frac{a_{21}\tilde{\xi}_1}{a_{11}a_{22}} \right).$$

We choose $\lambda = [d(p^\circ)]^{-2}$ and

$$a_{11} = \frac{\sqrt{\lambda \det(\partial_{ij}^2 \hat{u})}}{\sqrt{\partial_{22}^2 \hat{u}}}(p^\circ), \quad a_{21} = \frac{\sqrt{\lambda} \partial_{21}^2 \hat{u}}{\sqrt{\partial_{22}^2 \hat{u}}}(p^\circ), \quad a_{22} = \sqrt{\lambda \partial_{22}^2 \hat{u}}(p^\circ).$$

Denote by $\tilde{p}^\circ, \tilde{p}, \dots$ the image of p°, p, \dots under the affine transformation T . Then by a direct calculation we have $\tilde{\partial}_{ij}^2 \tilde{u}(\tilde{p}^\circ) = \delta_{ij}$, and for any p, q

$$\tilde{\partial}_2 \tilde{u}(\tilde{p}) = \frac{\lambda}{a_{22}} \partial_2 \hat{u}(p), \quad \tilde{\partial}_{22}^2 \tilde{u}(\tilde{p}) = \frac{\lambda}{a_{22}^2} \partial_{22}^2 \hat{u}(p), \quad \tilde{d}_{\tilde{u}}(\tilde{p}, \tilde{q}) = \sqrt{\lambda} d_u(p, q), \quad (6.29)$$

$$\lim_{k \rightarrow \infty} \max |\mathcal{S}(\tilde{u}_k)| = \lim_{k \rightarrow \infty} \max \frac{\mathcal{S}(u_k)}{\lambda_k} = \lim_{k \rightarrow \infty} \max d(p_k^\circ) \mathcal{S}(u_k) = 0, \quad (6.30)$$

where $\tilde{\partial}_i \tilde{u} = \frac{\partial \tilde{u}}{\partial \xi_i}$, $\tilde{\partial}_{ij}^2 \tilde{u} = \frac{\partial^2 \tilde{u}}{\partial \xi_i \partial \xi_j}$. In particular, $\tilde{d}(\tilde{p}^\circ) = 1$, and

$$\lim_{k \rightarrow \infty} \frac{\tau \tilde{d}(\tilde{p}_k^\circ)}{\int_{\tilde{\mathcal{B}}(\tilde{p}_k)} \sqrt{\partial_{22}^2 \tilde{u}_k} d\tilde{\xi}_2} = \lim_{k \rightarrow \infty} \frac{\tau d(\tilde{p}_k^\circ)}{\int_{\mathcal{B}(\tilde{p}_k)} \sqrt{\partial_{22}^2 u_k} d\xi_2} = 0, \quad (6.31)$$

where

$$\tilde{m}(\tilde{p}^\circ) = \max_{B_\tau(\tilde{p}^\circ)} |\tilde{\partial}_2 \tilde{u}|, \quad \tilde{\mathcal{B}}(\tilde{p}) = \{q \in \mathfrak{t}_2^* | |\tilde{\partial}_2 \tilde{u}|(q) \leq \tilde{m}(\tilde{p}^\circ)\}.$$

On the other hand, using Lemma 6.3 for $(\tilde{u}_k, \tilde{p}_k^\circ, \tilde{\tilde{p}}_k)$, we conclude that \tilde{u}_k locally uniformly converges to \tilde{u}_∞ and

$$\tilde{m}(\tilde{p}^\circ) \leq C_1^{-1}, \quad \tilde{\mathcal{B}}(\tilde{p}) \subset S_{\tilde{h}}(\tilde{p}, 1), \quad |\tilde{\xi}_2(q)| \leq C_1/2, \quad \forall q \in \tilde{\mathcal{B}}(\tilde{p}^\circ). \quad (6.32)$$

Then

$$\int_{\tilde{\mathcal{B}}(\tilde{p})} \sqrt{\partial_{22}^2 \tilde{u}_k} d\tilde{\xi}_2 \leq \left(\int_{\tilde{\mathcal{B}}(\tilde{p})} \partial_{22}^2 \tilde{u}_k d\tilde{\xi}_2 \right)^{\frac{1}{2}} \left(\int_{\tilde{\mathcal{B}}(\tilde{p})} d\tilde{\xi}_2 \right)^{\frac{1}{2}} \leq \tilde{m}(\tilde{p}^\circ) C_1 \leq 1.$$

It contradicts to (6.31). \square .

Proof of Proposition 6.4. If the Proposition is not true, by choosing a subsequence we can assume that

$$\lim_{k \rightarrow \infty} L(\Gamma_k) = 0, \quad (6.33)$$

where $L(\Gamma_k)$ denotes the geodesic arc-length of Γ_k . Moreover, we can assume that the Euclid measure of $\Gamma_k \cap \ell$ go to zero as $k \rightarrow \infty$. In fact, if the Euclid measure of $\Gamma_k \cap \ell$ has uniform positive lower bound, we can get a contradiction easily from $\partial_{22}^2 u|_{\ell_{c-\epsilon_o, d+\epsilon_o}} \geq C$.

There is an open set $U \subset \ell$ of $\Gamma \cap \ell_{c-\epsilon_o, d+\epsilon_o}$ such that $\ell_{c-\epsilon_o, d+\epsilon_o} \setminus U$ is a compact set and the Euclidean measure of U less than $\epsilon_o/2$, as k is large enough. Denotes

$$L_{c-2\epsilon_o} = \{(\xi_1, c-2\epsilon_o) | 0 \leq \xi_1 \leq \delta_1\}, \quad L_{d+2\epsilon_o} = \{(\xi_1, d+2\epsilon_o) | 0 \leq \xi_1 \leq \delta_1\}.$$

Let $\delta_1 > 0$ is a constant such that

$$\max_{L_{c-2\epsilon_o}} \partial_2 u \leq \min_{L_{c-\epsilon_o}} \partial_2 u, \quad \min_{L_{d+2\epsilon_o}} \partial_2 u \geq \max_{L_{d+\epsilon_o}} \partial_2 u. \quad (6.34)$$

Denote $p_1 = (0, c - \epsilon_2)$, $q_1 = (0, d + \epsilon_2)$.

Since the geodesic arc-length of Γ_k converges to zero as $k \rightarrow \infty$, by the interior regularity we can assume that $\Gamma \cap \{\xi \in E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_1} | \xi_1 = \delta_1\} = \emptyset$.

We discuss three cases:

Case 1. $\Gamma \cap L_{c-2\epsilon} \neq \emptyset$. Since $\partial_2 u$ is continuous on $\Gamma \cap E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_1}$, by (6.34) we have

$$[\partial_2 u(p_1), \partial_2 u(p)] \subset \bigcup_{\xi \in \Gamma \cap E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_1}} \partial_2 u(\xi).$$

Hence

$$\ell_{c-\epsilon_o, c} \setminus U \subset \bigcup_{p^\circ \in \Gamma \setminus \ell} \mathcal{B}(\check{p}), \quad |\ell_{c-\epsilon_o, c} \setminus U| \geq \epsilon_o/2.$$

Case 2. $\Gamma \cap L_{d+2\epsilon} \neq \emptyset$. By the same argument of Case 1, we have

$$\ell_{d, d+\epsilon_o} \setminus U \subset \bigcup_{p^\circ \in \Gamma \setminus \ell} \mathcal{B}(\check{p}), \quad |\ell_{d, d+\epsilon_o} \setminus U| \geq \epsilon_o/2.$$

Case 3. $\Gamma \subset E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_1}$. Since $\partial_2 u$ is continuous on Γ , we have

$$[\partial_2 u(p), \partial_2 u(q)] \subset \bigcup_{\xi \in \Gamma \cap E_{c-2\epsilon_o, d+2\epsilon_o}^{\delta_1}} \partial_2 u(\xi).$$

Hence

$$\ell_{c, d} \setminus U \subset \bigcup_{p^\circ \in \Gamma \setminus \ell} \mathcal{B}(\check{p}), \quad |\ell_{c, d} \setminus U| \geq \epsilon_o/2$$

as k is large enough.

We prove the Case 3 (the proof of the other cases is the same). There are finite many points $\{p_k^\circ\}_1^N$ such that $\{\mathcal{B}(\check{p}_k)\}_1^N$ covers $\ell_{c, d} \setminus U$ and

$$\mathcal{B}(\check{p}_i) \cap \mathcal{B}(\check{p}_j) \cap \mathcal{B}(\check{p}_k) = \emptyset,$$

for any different i, j, k . Then

$$B_{\tau d_i}(p_i^\circ) \cap B_{\tau d_j}(p_j^\circ) \cap B_{\tau d_k}(p_k^\circ) = \emptyset,$$

where $d_i = d(p_i^\circ, \partial\Delta)$. In fact, if there exists a point $p^* \in B_{\tau d_i}(p_i^\circ) \cap B_{\tau d_j}(p_j^\circ) \cap B_{\tau d_k}(p_k^\circ)$, let $q^* \in \ell$ such that $\partial_2 u(q^*) = \partial_2 u(p^*)$. By definition of $\mathcal{B}(\check{p}_k)$

we have $q^* \in \mathcal{B}(\check{p}_i) \cap \mathcal{B}(\check{p}_j) \cap \mathcal{B}(\check{p}_k)$. We get a contradiction. Therefore $L(\Gamma) \geq \frac{1}{2} \sum_i^N L(p_i^\circ)$. By Lemma 6.2 and Lemma 6.5 we have

$$L(\Gamma) \geq \sum_1^N \tau d_i \geq \frac{1}{C_2} \int_{\ell_{c,d} \setminus U} \sqrt{\partial_{22}^2 u} d\xi_2 \geq \frac{\sqrt{C_7} \epsilon_o}{C_2}.$$

It contradicts to (6.33). We finish the proof of Proposition 6.4. \square .

Let L_c and L_d be its upper and lower edges,

$$L_c = \{(\xi_1, c) | 0 \leq \xi_1 \leq \delta'\}, \quad L_d = \{(\xi_1, d) | 0 \leq \xi_1 \leq \delta'\}.$$

By the same argument, we have

Proposition 6.6 *There exists a constant $C_9 > 0$ such that*

$$d_{u_k}(p, q) > C_9$$

for $p \in L_c$ and $q \in L_d$.

By the interior regularities and the same argument of Proposition 6.4, we conclude that

Theorem 6.7 *For any $p \in \ell_{c,d}$ there exists a small constant ϵ , independent of k , such that the intersection of the geodesic ball $B_\epsilon^{(k)}(p)$ and Δ is contained in a Euclidean half-disk $D_a(p) \cap \Delta$ for some $a > 0$ with $D_a(p) \cap \partial\Delta \subset \ell_{c-\epsilon_o, d+\epsilon_o}$.*

7 Upper Bound of H

The following theorem has been proved in [11].

Theorem 7.1 *Let (M, G) be a compact complex manifold of dimension n with Kähler metric G . Let ω_o be its Kähler form. Denote*

$$\mathcal{C}^\infty(M, \omega_o) = \{\phi \in C^\infty(M) | \omega_\phi = \omega_o + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0\}.$$

Then for any $\phi \in \mathcal{C}^\infty(M, \omega_o)$, we have

$$H \leq \left(2 + \frac{\max_M |\mathcal{S}(f)|}{n\dot{\mathcal{K}}}\right)^n \exp \left\{2\dot{\mathcal{K}}(\max_M \{\phi\} - \min_M \{\phi\})\right\}. \quad (7.1)$$

where $f = g + \phi$ and $\dot{\mathcal{K}} = \max_M \|Ric(g)\|_g^2$, $Ric(g)$ denotes Ricci tensors of the metric ω_g .

8 Lower bound of H

In this section we will prove the following theorem.

Theorem 8.1 *Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope and (M, ω_o) be the associated compact toric surface. Let $K \in C^\infty(\bar{\Delta})$ be an edge-nonvanishing function and $u_k = v + \psi_k \in \mathcal{C}^\infty(\Delta, v)$ be a sequence of functions with $\mathcal{S}(u_k) = K_k$. Suppose that*

(1) K_k converges to K smoothly on $\bar{\Delta}$;

(2) $\max_{\bar{\Delta}} |u_k| \leq C_1$,

where C_1 is a constant independent of k . Then there exist a constant $C_{10} > 1$ independent of k such that for any k

$$C_{10}^{-1} \leq H_{f_k} \leq C_{10}. \quad (8.1)$$

The upper bound is proved. Now suppose that there is no positive lower bound. Then by choosing a subsequence we can assume that $\min_M(H_{f_k}) \rightarrow 0$. Let $p_k \in \bar{\Delta}$ be the minimum point of H_{f_k} , that is, for any $z_k \in \tau_{f_k}^{-1}(p_k)$, $H_{f_k}(z_k) = \min_M H_{f_k}$. Let p_∞ be the limit of p_k (if necessary, by taking a subsequence to get the limit.) Then by the interior regularity theorem (cf. Theorem 3.8), we can assume that $p_\infty \in \partial\Delta$.

Let f_k and p_k be as above. Let \mathcal{N} be a large constant to be determined (cf. (8.12)). For any point $q \in \bar{\Delta}$ with the property

$$H_{f_k}(q) \leq \mathcal{N} H_{f_k}(p_k)$$

we say that q is an H -minimal equivalence point for f_k . For any two such points q_1 and q_2 we write $q_1 \stackrel{min}{\sim} q_2$.

Similarly, for any sequence q_k of H -minimal equivalence points for f_k , the limit of q_k must be in $\partial\Delta$.

8.1 A subharmonic function and seeking minimal equivalence points

Let \mathbb{C}_ϑ^2 be a coordinate chart associated to the vertex ϑ . Let \mathcal{Q}_ϑ be the space of coordinates of radius of \mathbb{C}_ϑ^2 . It is the first quadrant of \mathbb{R}^2 . We omit the

index ϑ if there is no danger of confusion. We have a natural map

$$\rho : \mathbb{C}^2 \rightarrow \mathcal{Q}, \quad \rho(z_1, z_2) = (r_1, r_2) = (|z_1|, |z_2|).$$

Since we consider \mathbb{T}^2 -invariant objects, we identify \mathbb{C}^2 as \mathcal{Q} in the following sense: when we write a set $\Omega \subset \mathcal{Q}$, we mean $\rho^{-1}(\Omega)$. We also note that \mathcal{Q}° (the interior of \mathcal{Q}) is identified with \mathfrak{t} in a canonical way: $x_i = 2 \log r_i$.

Introduce the notations in \mathcal{Q}

$$\text{Box}(a; b) = \{(r_1, r_2) | r_1 \leq a, r_2 \leq b\}.$$

$$\text{Box}(a_1, a_2; b_1, b_2) = \{a_1 \leq r_1 \leq a_2, b_1 \leq r_2 \leq b_2\}.$$

Let $\mathcal{B}_\vartheta = \text{Box}(1, 1)$ in \mathcal{Q}_ϑ . Its boundary consists of two parts, $E_\vartheta^i = \{(r_1, r_2) | |r_i| = 1, |r_{3-i}| \leq 1\}, i = 1, 2$. (Here, by the boundary we mean the boundary of $\rho^{-1}(\mathcal{B})$ in \mathbb{C}^2 , hence the boundaries of the box located on the axis are in fact the interior of the complex manifold.)

For a toric surface, we have the following simple lemma.

Lemma 8.2 (i) All \mathcal{B}_ϑ 's in M intersect at one point, i.e, $(1, 1)$ in each \mathcal{B}_ϑ ;
(ii) For any two vertices ϑ and ϑ° next to each other, \mathcal{B}_ϑ and $\mathcal{B}_{\vartheta^\circ}$ share a common boundary; (iii) $M = \bigcup_\vartheta \mathcal{B}_\vartheta$.

Proof. Let ϑ and ϑ° be two vertices that are next to each other. Let ℓ be the edge connecting them. We put Δ in the first quadrant of \mathfrak{t}^* as the following: (1) ϑ at the origin; (2) ℓ on the ξ_2 -axis; (3) the other edge ℓ_* of ϑ on the ξ_1 -axis; (4) suppose that $\vartheta^\circ = (0, c_o)$ and its other edge ℓ^* is given by the equation $\xi_2 = a_o \xi_1 + c_o$ for some integer a_o .

The edges ℓ_* and ℓ is of the form $\vartheta + te_1$ and $\vartheta + te_2, t \in \mathbb{R}$ respectively. Here $\{e_1, e_2\}$ is a basis of \mathbb{Z}^2 . Similarly, ℓ^* and ℓ is of the form $\vartheta^\circ + te_1^\circ$ and $\vartheta^\circ + te_2^\circ, t \in \mathbb{R}$ respectively, and $\{e_1^\circ, e_2^\circ\}$ is a basis of \mathbb{Z}^2 . Then

$$e_1^\circ = e_1 + a_o e_2, \quad e_2^\circ = -e_2.$$

For any point $p \in \Delta$, we have

$$p = (e_1, e_2)(\xi_1, \xi_2)^t = (e_1^\circ, e_2^\circ)(\xi_1^\circ, \xi_2^\circ)^t + (0, c_o)^t,$$

where A^t denotes the transpose of a matrix A . Hence the coordinate transformation between (ξ_1, ξ_2) and $(\xi_1^\circ, \xi_2^\circ)$ is

$$\xi_1^\circ = \xi_1, \quad \xi_2^\circ = a_o \xi_1 - \xi_2 + c_o,$$

and the coordinate transformation between (x_1, x_2) and (x_1^o, x_2^o) is

$$x_1^o = x_1 + a_o x_2, \quad x_2^o = -x_2.$$

Then the coordinate transformation between \mathbb{C}_ϑ^2 and $\mathbb{C}_{\vartheta^o}^2$ is given by

$$z_1^o = z_1 z_2^{a_o}, \quad z_2^o = z_2^{-1}. \quad (8.2)$$

By this, we find that \mathcal{B}_ϑ and $\mathcal{B}_{\vartheta^o}$ intersect at the common boundary $E_\vartheta^2 = E_{\vartheta^o}^2$. The rest of the facts of the lemma can be derived easily as well. q.e.d.

Let \mathcal{E} be the collection of all E_ϑ^i 's.

Now consider an element $\mathbf{f} \in \mathcal{C}^\infty(M, \omega; K_o)$. Let f_ϑ be its restriction to \mathcal{U}_ϑ . We introduce a *subharmonic* function

$$F_\vartheta = \log W_\vartheta + N f_\vartheta.$$

Lemma 8.3 *If $N \geq \max |\mathcal{S}(f_\vartheta)| + 1$, $\square F_\vartheta > 0$. Hence the maximum of F_ϑ on \mathcal{B}_ϑ is achieved on $E_\vartheta^1 \cup E_\vartheta^2$.*

Proof. By a direct computation,

$$\square F_\vartheta = -\mathcal{S}(f_\vartheta) + 2N > 0.$$

q.e.d.

Lemma 8.4 *All F_ϑ on \mathcal{B}_ϑ form a continuous function \mathbf{F} on M .*

Proof. Let ϑ and ϑ^o be two vertices that are next to each other as in the proof of Lemma 8.2. By a direct calculation, we have

$$\begin{aligned} f_{\vartheta^o} - f_\vartheta &= \sum (x_i^o \xi_i^o - u) - \sum (x_i \xi_i - u) \\ &= \sum x_i^o \xi_i^o - (x_1^o + a_o x_2^o) \xi_1^o - (-x_2^o)(a_o \xi_1^o - \xi_2^o + c_o) \\ &= c_o \log |r_2^o|^2. \end{aligned} \quad (8.3)$$

$$\log W_{\vartheta^o} - \log W_\vartheta = \log \left| \frac{\partial z_i}{\partial z_j^o} \right|^2 = (a_o - 2) \log |r_2^o|^2. \quad (8.4)$$

From this we conclude that F_{ϑ^o} and F_ϑ match on their common boundary (where $r_2 = r_2^o = 1$). Hence, all F_ϑ 's form a function on M . q.e.d

Hence \mathbf{F} is a continuous function on M and piecewise subharmonic.

Definition 8.5 A point $p \in E \subset \mathcal{E}$ is called a local maximum point of F if it is a maximum in $\mathcal{B}_\vartheta \cup \mathcal{B}_{\vartheta^o}$, where the vertices ϑ, ϑ^o are chosen such that $\mathcal{B}_\vartheta \cap \mathcal{B}_{\vartheta^o} = E$.

Corollary 8.6 The set of local maximum points of F is nonempty.

Recall that $W_{g_\vartheta} = \det((g_\vartheta)_{i\bar{j}})$. We now use F to construct an H -minimal equivalence point.

Lemma 8.7 There is a local maximum point q_o of F that is an H -minimal equivalence point.

Proof. Let z_o be the minimal point of H . Suppose that it is in $\mathcal{B}_{\vartheta_0}$ for some vertex ϑ_0 . Suppose that F_{ϑ_0} achieves its maximum at p_0 . We claim that $p_0 \stackrel{\min}{\sim} z_o$: in fact, by the assumption, $F_{\vartheta_0}(p_0) \geq F_{\vartheta_0}(z_o)$. Explicitly, this is $\log W_{g_{\vartheta_0}}(p_0) + Nf_{\vartheta_0}(p_0) \geq \log W_{g_{\vartheta_0}}(z_o) + Nf_{\vartheta_0}(z_o)$. Hence, by the definition of H , we have

$$\log H(p_0) - \log W_{g_{\vartheta_0}}(p_0) - Nf_{\vartheta_0}(p_0) \leq \log H(z_o) - \log W_{g_{\vartheta_0}}(z_o) - Nf_{\vartheta_0}(z_o).$$

In $\mathcal{B}_{\vartheta_0}$, $|f_{\vartheta_0}|$ and $|\log W_{g_{\vartheta_0}}|$ are uniformly bounded. Therefore, there exists a constant C such that

$$\log H(p_o) \leq \log H(z_o) + C.$$

This implies the claim.

Note that p_0 is in the boundary of $\mathcal{B}_{\vartheta_0}$. Suppose that it is contained in E which is the boundary of the other box $\mathcal{B}_{\vartheta_1}$. If p_0 is also the maximal point for F_{ϑ_1} , it is a local maximal point. The problem is solved. Otherwise, we consider the maximal point p_1 of F_{ϑ_1} . It should be on the other edge of $\mathcal{B}_{\vartheta_1}$. If it is a local maximum, we are done. Otherwise, we repeat the argument. Of course, this will stop in a finite number of steps. q.e.d.

Remark 8.8 Let $q_o \in E$ be a point given by Lemma 8.7. We suppose that $E = \mathcal{B}_\vartheta \cap \mathcal{B}_{\vartheta^o}$. We use the local models and notations described in the proof of Lemma 8.2.

Let

$$A_\vartheta^\delta = \text{Box}(\delta, \delta^{-1}; \delta, \delta^{-1}) \subset \mathcal{Q}_\vartheta.$$

Set $M^\delta \subset M$ to be the union of all A_ϑ^δ . For simplicity we choose $\delta = \frac{1}{1000}$. According to the explanation at the beginning of the section, we make the following assumption.

Assumption 8.9 Any H -minimal equivalence point is not in M^δ .

8.2 A technical lemma

The main purpose of this subsection is to prove the following lemma.

Lemma 8.10 *Let f be a function in the sequence f_k that satisfies Assumption 8.9. Let q_o be the point in Lemma 8.7 (we use the notations in Remark 8.8). Let*

$$I^+ = \{(0, r_2) | 1 \leq r_2 \leq e\}, \quad I^- = \{(0, r_2) | e^{-1} \leq r_2 \leq 1\},$$

$$J^+ = \{(0, r_2) | 1 \leq r_2 \leq e/2\}, \quad J^- = \{(0, r_2) | 2(e)^{-1} \leq r_2 \leq 1\}$$

and $I = I^+ \cup I^-$. Then there is a constant $C_{11} > 0$ independent of k such that either one of the following cases holds:

- (1) $\max_{q \in I^+ \setminus J^+} H(q) \leq C_{11} \min_M H$, namely, for any $q \in I^+ \setminus J^+$, $q \stackrel{\min}{\sim} q_o$;
- (2) $\max_{q \in I^- \setminus J^-} H(q) \leq C_{11} \min_M H$, namely, for any $q \in I^- \setminus J^-$, $q \stackrel{\min}{\sim} q_o$;
- (3) for any $(0, s) \in J^+$, there is a point $r(s)$ with $r_1(s) \leq \delta, r_2(s) = s$ such that $r(s) \stackrel{\min}{\sim} q_o$ and

$$\max_I H \leq C_{11} \max_{q \in \Gamma_s} H(q),$$

where Γ_s is the segment $\{(r_1, r_2) | r_1 \leq r_1(s), r_2 = s\}$.

First we explain the main idea of the proof of Lemma 8.10. By a coordinate transformation, we may identify \mathcal{Q}_ϑ and $\mathcal{Q}_{\vartheta^o}$. For simplicity, we assume that $a_o = 0$ (cf. (8.2)) and the coordinate transformation is

$$z_1^o = z_1, z_2^o = z_2^{-1}.$$

Then we have an identification between $\mathcal{Q}_{\vartheta^o}$ and \mathcal{Q}_ϑ by $r_1^o = r_1, r_2^o = r_2^{-1}$.

Step 1. We will construct a curve L given by $\eta : \mathbb{R} \rightarrow \Omega = \{0 \leq r_1 \leq 1/2\}$ such that

- 1. $\log r_2(\eta(t)) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$;
- 2. there exist positive constants N_1 and C_0 independent of k such that for any t

$$F_\vartheta(\eta(t)) + N_1 |\log r_2(\eta(t))| + C_0 \geq F_\vartheta(q_o), \quad (8.5)$$

$$F_{\vartheta^o}(\eta(t)) + N_1 |\log r_2^o(\eta(t))| + C_0 \geq F_{\vartheta^o}(q_o) = F_\vartheta(q_o) \quad (8.6)$$

Here $F_{\vartheta^o}(q_o) = F_\vartheta(q_o)$ follows from Lemma 8.4.

3. L , the r_1 -axis and the r_2 -axis bound a connected domain in Ω . We denote the domain by Ω_\star .

We postpone the construction. We give the proof of Lemma 8.10 by assuming the existence of such a curve L .

Step 2. For any $m > 1$ large and $(0, b) \in J^+$, define

$$\tilde{\Omega}_\star^+(b, m) = \{(r_1, r_2) \in \Omega_\star | b \leq r_2 \leq m\},$$

$$\tilde{\Omega}_\star^-(b, m) = \{(r_1, r_2) \in \Omega_\star | m^{-1} \leq r_2 \leq b\}.$$

Let $\Omega_\star^+(b, m)$ (resp. $\Omega_\star^-(b, m)$) be the component that contains $\{(0, r_2) | b \leq r_2 \leq m\}$ (resp. $\{(0, r_2) | m^{-1} \leq r_2 \leq b\}$).

Consider the function

$$P_b = \log W_{\vartheta} - N f_{\vartheta} + \alpha \log r_2^2 + \beta |\log r_2^2 - \log b^2|,$$

$$P_b^o = \log W_{\vartheta^o} - N f_{\vartheta^o} + \alpha \log (r_2^o)^2 + \beta |\log (r_2^o)^2 - \log b^2|.$$

By the calculation of Lemma 8.3 and the choosing of N , it is easy to see that in $\Omega_\star^+(b, m)$ and $\Omega_\star^-(b, m)$

$$\square P_b < 0. \tag{8.7}$$

In order to have $P_b = P_b^o$, we let α satisfy

$$(a_0 - 2) - N c_0 + 2\alpha = 0.$$

$\beta > |\alpha|$ is a large constant to be determined later(cf. (8.10)).

Proof of Lemma 8.10. We apply the minimum principle to P_b on $\Omega_\star^+(b, m)$ and $\Omega_\star^-(b, m)$. Then the boundary of $\Omega_\star^+(b, m)$ consists of three parts:

$$\partial_{1,b}^+ \subset L, \quad \partial_{2,b}^+ \subset \{r_2 = b\}, \quad \partial_3^+ \subset \{r_2 = m\}.$$

Similarly, the boundary of $\Omega_\star^-(b, m)$ consists of three parts:

$$\partial_{1,b}^- \subset L, \quad \partial_{2,b}^- \subset \{r_2 = b\}, \quad \partial_3^- \subset \{r_2 = m^{-1}\}.$$

By (8.7) we conclude that P_b cannot attain its minimum at an interior point in $\Omega_\star^+(b, m)$ (resp. $\Omega_\star^-(b, m)$).

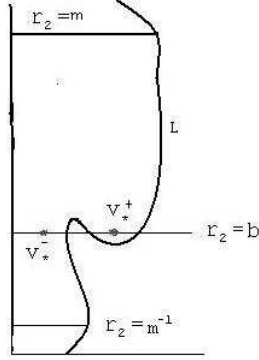


Figure 1:

We consider P_b^o in $\Omega_\star^+(b, m)$. Noting that $\Omega_\star^+(b, m) \subset \{r_1^o \leq \frac{1}{2}, r_2^o \leq b^{-1}\}$, we have $\log W_{g_{\vartheta^o}}$ has uniform bound independent of m . By Theorem 7.1 we obtain that in $\Omega_\star^+(b, m)$,

$$\log W_{\vartheta^o} = \log W_{g_{\vartheta^o}} - \log H \geq -C_0$$

for some constant $C_0 > 0$ independent of m , where we used Theorem 7.1. Combing this and $|f_{\vartheta^o} - g_{\vartheta^o}| \leq \mathcal{C}_1$ we have $\log W_{\vartheta^o} - Nf_{\vartheta^o} \geq -C'_1$ in $\Omega_\star^+(b, m)$. When $\beta > |\alpha|$, a direct calculation give us $\lim_{r_2^o \rightarrow 0} P_b^o = +\infty$. By the same argument we have $\lim_{r_2 \rightarrow 0} P_b = +\infty$. Since $P_b = P_b^o$, by choosing m large (this is allowed to depend on \mathbf{f}), we can assume that the minimum point of P_b cannot be on ∂_3^+ (resp. ∂_3^-).

We discuss three cases.

Case 1. There is $(0, b) \in J^+$ such that on $\Omega_\star^+(b, m)$, P_b achieves its minimum at $v_*^+ \in \partial_{1,b}^+$. Then for any $q \in I^+ \setminus J^+$, $P_b(q) \geq P_b(v_*^+)$. On the other hand,

$$F_{\vartheta}(v_*^+) + N_1 \log r_2(v_*^+) + C_0 > F_{\vartheta}(q_o). \quad (8.8)$$

Since $x_1(v_*^+) \frac{\partial f_{\vartheta}}{\partial x_1}(v_*^+) \leq 0$, $x_2(v_*^+) > 0$, $|u| \leq \mathcal{C}_1$ and $\left| \frac{\partial f_{\vartheta}}{\partial x_2} \right| \leq C_2$ with $C_2 = \text{diam}(\Delta)$, we have, at v_*^+ ,

$$f_{\vartheta} = \sum x_i \frac{\partial f_{\vartheta}}{\partial x_i} - u \leq C_2 x_2 + \mathcal{C}_1. \quad (8.9)$$

Choose

$$\beta = |\alpha| + 2NC_2 + 2N_1 + 1. \quad (8.10)$$

Note that $P_b = F_\vartheta - 2Nf_\vartheta + (\alpha + \beta)x_2 - \beta \log b^2$, which together with (8.8) and (8.9) give us

$$F_\vartheta(q) \geq F_\vartheta(q_o) - C_3, \quad (8.11)$$

where $C_3 > 1$ is a constant independent of k . Choose

$$\mathcal{N} = e^{C_3}(N')^2, \quad (8.12)$$

where

$$N' = \exp \left\{ 4 \sum_{\vartheta} \left(\max_{\mathcal{B}_\vartheta} |\log W_{g_\vartheta}| + \max_{\mathcal{B}_\vartheta} N(|g_\vartheta| + \mathcal{C}_1) \right) \right\}. \quad (8.13)$$

Since $|f_\vartheta|$ and $|\log W_{g_\vartheta}|$ are uniformly bounded at q_o and in I^+ , (8.11) says that $q \stackrel{\min}{\sim} q_o$. (1) is proved.

Case 2. There is $(0, b) \in J^+$ such that on $\Omega_\star^-(b, m)$, P_b achieves its minimum at $v_\star^- \in \partial_{1,b}^-$. Since $P_b = P_b^\circ$, we consider the function P_b° in $\Omega_\star^-(b, m)$. By the same argument as Case 1 we can prove (2).

Case 3. For any $(0, b) \in J^+$, the minimum of P_b on $\Omega_\star^+(b, m)$ is achieved at $v_\star^+(b) \in \partial_{2,b}^+$ and the minimum of P_b on $\Omega_\star^-(b, m)$ is achieved at $v_\star^-(b) \in \partial_{2,b}^-$.

For any $(0, s) \in J^+$, set $r_2(s) = s$ and let $r(s) \in L$ be the point such that for any $(\tilde{r}_1, s) \in L$, $r_1(s) \geq \tilde{r}_1$. Obviously, $r_1(s) \geq r_1(v_\star^+(s))$ and $r_1(s) \geq r_1(v_\star^-(s))$. Since $f_\vartheta, f_{\vartheta^\circ}, \log r_2, \log r_2^\circ, \log W_{g_\vartheta}$ and $\log W_{g_{\vartheta^\circ}}$ are uniformly bounded in $\{r_1 \leq \frac{1}{2}, e^{-1} \leq r_2 \leq e\}$. Then from

$$P_b = \log W_{g_\vartheta} - \log H - Nf_\vartheta + \alpha \log r_2^2 + \beta |\log r_2^2 - \log b^2|$$

$$P_b^\circ = \log W_{g_{\vartheta^\circ}} - \log H - Nf_{\vartheta^\circ} + \alpha \log(r_2^\circ)^2 + \beta |\log(r_2^\circ)^2 + \log b^2|.$$

we derive (3) of Lemma 8.10. q.e.d.

Construction of L in Step 1. Let $\chi(b)$ be the maximum point of F_ϑ in $\text{Box}(\frac{1}{2}; b)$. Note that $F_\vartheta(q_o) = \max_{\mathcal{B}_\vartheta} F_\vartheta$. Then by the definition, we have

$$\chi(1) = q_o, \quad F_\vartheta(\chi(\frac{1}{2})) < F_\vartheta(q_o).$$

Let $\alpha \geq F_\vartheta(q_o) - 1$ be a *regular value* of F_ϑ and be between $F_\vartheta(\chi(\frac{1}{2}))$ and $F_\vartheta(q_o)$. Set $\tilde{U} = F_\vartheta^{-1}([\alpha, \infty))$ and $U = \tilde{U} \cap \Omega$. Let U' be the component of U containing q_o .

Lemma 8.11 $\partial U'$ either goes to $r_2 = \infty$ or hits $\{r_1 = \frac{1}{2}\}$.

Proof. If Lemma is not true, there are two positive constants C, ϵ_1 such that for any $q \in U'$,

$$r_1(q) \leq \frac{1}{2} - \epsilon_1, \quad \frac{1}{2} \leq r_2(q) \leq C.$$

Since F_ϑ is continuous, by the definition of U' we conclude that $F_\vartheta|_{\partial U'} = \alpha$. By the maximal principle of the subharmonic function, we have

$$F_\vartheta(q) = \alpha \quad \text{and} \quad \square F(q) = 0, \quad \forall q \in U'.$$

On the other hand, F_ϑ is a strictly subharmonic function, we get a contradiction. q.e.d.

We are now ready to construct a curve \tilde{L}^+ given by $\tilde{\eta}^+ : [0, \infty) \rightarrow U'$ as the following:

- (1). $\tilde{\eta}^+(0) = q_o$;
- (2). if the boundary $\partial U'$ meets $\{r_1 = \frac{1}{2}\}$, let $q^+ \in \partial U' \cap \{r_1 = \frac{1}{2}\}$. We let $\tilde{\eta}^+([0, 2]) \subset U'$ be a curve satisfying $\tilde{\eta}^+(0) = q_o$, $\tilde{\eta}^+(2) = q^+$. Notice that, by the assumption 8.9, $r_2(q^+) \geq \delta^{-1}$;
- (3). continue (2), suppose $q^+ = (\frac{1}{2}, s)$, then let

$$\tilde{\eta}^+(t) = \left(\frac{1}{2}, s + (t - 2) \right), t \geq 2;$$

- (4). if the boundary $\partial U'$ goes to $r_2 = \infty$, let $q^\infty \in \partial U' \cap \{r_2 = \infty\}$. We let $\tilde{\eta}^+([0, +\infty)) \subset U'$ be a curve satisfying $\tilde{\eta}^+(0) = q_o$, $\tilde{\eta}^+(\infty) = q^\infty$.

Symmetrically, we may consider F_{ϑ^o} on Ω and construct another curve \tilde{L}^- defined in $(-\infty, 0]$. Then it matches \tilde{L}^+ at the starting point q_o and goes to $\{r_2 = 0\}$. Let \tilde{L} be the union of \tilde{L}^+ and \tilde{L}^- given by $\tilde{\eta}$. Thus $\tilde{\eta}(t) = \tilde{\eta}^+(t)$, $\forall t \geq 0$ and $\tilde{\eta}(t) = \tilde{\eta}^-(t)$, $\forall t \leq 0$. Then by the construction,

$$\lim_{t \rightarrow +\infty} \log r_2(\tilde{\eta}^+(t)) = +\infty, \quad \lim_{t \rightarrow -\infty} \log r_2(\tilde{\eta}^-(t)) = -\infty.$$

It is possible that \tilde{L} has self-intersections. We can modify it such that the new curve L (the part of \tilde{L}) has no self-intersections. For example, if $\tilde{\eta}$ has only a self-intersection with $\tilde{\eta}(t_1) = \tilde{\eta}(t_2)$, $t_1 < t_2$, then we let

$$\eta(t) = \tilde{\eta}(t_1), \quad \forall t \in [t_1, t_2], \quad \eta(t) = \tilde{\eta}(t), \quad \forall t \in (-\infty, t_1) \cup (t_2, +\infty).$$

Denote the curve $\eta(t)$ by L . Then

$$\lim_{t \rightarrow +\infty} \log r_2(\eta(t)) = +\infty, \quad \lim_{t \rightarrow -\infty} \log r_2(\eta(t)) = -\infty.$$

It is possible that $\tilde{L}^+ \cap \tilde{L}^- \neq \emptyset$; in this case it is possible that $q_o \notin L$, but we still have

$$\eta(0) \in \tilde{L}^+ \cap \tilde{L}^- \subset \{(r_1, r_2) | r_1 \leq \frac{1}{2}, \frac{1}{2} \leq r_2 \leq 2\}.$$

Since $|f_\vartheta|$ and $|\log W_{g_\vartheta}|$ are uniformly bounded in $\{(r_1, r_2) | r_1 \leq \frac{1}{2}, \frac{1}{2} \leq r_2 \leq 2\}$, $\eta(0) \stackrel{\min}{\sim} q_o$. For simplicity, we assume that $\eta(0) = q_o$.

We only need to prove (8.5) and (8.6).

Lemma 8.12 *There exist positive constants C_0 and N_1 independent of k such that for any $q \in L$*

$$F_\vartheta(q) + N_1 |\log r_2(s)| + C_0 \geq F_\vartheta(q_o). \quad (8.14)$$

Proof. By the construction of \tilde{L}^+ , for any point q that were constructed in steps (2) and (4) we have

$$F_\vartheta(q) \geq F_\vartheta(q_o) - 1.$$

The claim is obvious except for the points that were constructed in step (3).

We now prove for $q = \tilde{\eta}^+(t), t \geq 2$, that

$$F_\vartheta(q) = \log W_\vartheta(q) + N f_\vartheta(q) = -\log \det(\partial_{ij}^2 u) - 2 \log r_1 - 2 \log r_2 + N f_\vartheta(q) + \log 4.$$

Since $r_1(q) = \frac{1}{2}$, $\det(\partial_{ij}^2 u) \geq C_4$, we conclude from (8.9) that $F_\vartheta(s) \leq C \log r_2 + C'$. In particular, by $F_\vartheta(q^+) \geq \alpha$ we have

$$F_\vartheta(q_o) \leq F_\vartheta(q^+) + 1 \leq C \log |r_2(q^+)| + C'. \quad (8.15)$$

On the other hand, by (8.3) and (8.4) we have

$$F_\vartheta(q) = \log W_{\vartheta^o}(q) - 2(Nc_o - 2) \log r_2^o(q) + N f_{\vartheta^o}(q).$$

Using the upper bound of H and uniform bound of W_g^o , we have

$$\log W_{\vartheta^o}(q) = \log W_{g_{\vartheta^o}}(q) - \log H \geq -C'.$$

Therefore,

$$F_{\vartheta}(q) + N'' \log r_2(q) + C'' \geq 0 \quad (8.16)$$

where N'' and C'' are positive constants independent of k . Combining (8.15) and (8.16), we prove that for any $q \in \tilde{L}^+$,

$$F_{\vartheta}(q) + (N'' + C)|\log r_2(q)| + C'' + C' + 1 \geq F_{\vartheta}(q_o). \quad (8.17)$$

By the same argument we can obtain that for any $q \in \tilde{L}^-$

$$F_{\vartheta^o}(q) + (N'' + C)|\log r_2^o(q)| + C'' + C' + 1 \geq F_{\vartheta^o}(q_o). \quad (8.18)$$

Choosing $N_1 = 2N'' + 2C + 4(Nc_o - 2)$ and $C_0 = 2 + 2C' + 2C''$. Note that $F_{\vartheta^o} = F_{\vartheta} + 2(Nc_o - 2) \log r_2^o$ and $|\log r_2^o| = |\log r_2|$. We prove (8.14) for $s \in L^-$. q.e.d.

Hence Step 1 is finished and Lemma 8.10 is proved.

Remark 8.13 For any $b_1 > b_2 > c > a_2 > a_1 > 0$, let

$$\tilde{I}^+ = \{(0, r_2) | c \leq r_2 \leq b_1\}, \quad \tilde{I}^- = \{(0, r_2) | a_1 \leq r_2 \leq c\},$$

$$\tilde{J}^+ = \{(0, r_2) | c \leq r_2 \leq b_2\}, \quad \tilde{J}^- = \{(0, r_2) | a_2 \leq r_2 \leq c\}.$$

By the same argument as in Lemma 8.10, we can prove that there is a constant $C_{12} > 0$ independent of k such that either one of the following cases holds:

$$(1) \quad \max_{q \in \tilde{I}^+ \setminus \tilde{J}^+} H(q) \leq C_{12} \min_M H;$$

$$(2) \quad \max_{q \in \tilde{I}^- \setminus \tilde{J}^-} H(q) \leq C_{12} \min_M H;$$

(3) for any $(0, s) \in \tilde{J}^+$, there is a point $r(s)$ with $r_1(s) \leq \delta, r_2(s) = s$ such that $r(s) \stackrel{\min}{\sim} q_o$

$$\max_{\tilde{I}^+ \cup \tilde{I}^-} H \leq C_{12} \max_{q \in \Gamma_s} H(q),$$

where Γ_s is the segment $\{(r_1, r_2) | r_1 \leq r_1(s), r_2 = s\}$.

8.3 Proof of Theorem 8.1

To prove Theorem 8.1 we need the following lemma, the proof of which is the same as Lemma 7.16 in [9].

Lemma 8.14 *Let $z^* \in Z$. Let f be a function in the sequence f_k that satisfies Assumption 8.9. Suppose that in $B_{2a}(z^*)$, $\mathcal{K} \leq C_0$ for some constant $C_0 > 0$ independent of k . Then there is a constant $c > 0$, independent of k , such that there is a point z^o in $B_a(z^*)$ satisfying*

$$d(z^o, B_{2a}(z^*) \cap Z) = c.$$

Obviously $c \leq a$. Hence $d(z^o, Z) = c$.

Proof of Theorem 8.1. The upper bound is proved in Theorem 7.1. Now we suppose that there is no positive lower bound. By choosing a subsequence we can assume that $\min_M(H_{f_k}) \rightarrow 0$. Let f be a function in the sequence f_k that satisfies Assumption 8.9. Let q_o be the point in Lemma 8.7. Let ϑ and ϑ^o be two vertices next to each other such that $q_o \in \mathcal{B}_\vartheta \cap \mathcal{B}_{\vartheta^o}$. Let ℓ be the edge connecting ϑ and ϑ^o . Choose the coordinate in \mathfrak{t}^* as the following: (1) ϑ at the origin; (2) ℓ on the ξ_2 -axis; (3) the other edge ℓ_* of ϑ on the ξ_1 -axis; (4) $\Delta \subset \{\xi | \xi_1 \geq 0, \xi_2 \geq 0\}$.

Let $\xi^{(\ell)} \in \ell^o$ and $\mathcal{D}^\ell := D_\epsilon(\xi^{(\ell)}) \cap \bar{\Delta}$ be a half ϵ -disk such that

$$d_E(\mathcal{D}^\ell, \partial\Delta \setminus \ell) > \delta_o, \quad |K| > \delta_o > 0, \quad \text{on } \mathcal{D}^\ell.$$

for some $\delta_o > 0$ independent of k . By $d_E(\mathcal{D}^\ell, \partial\Delta \setminus \ell) > \delta_o$ and $\max_{\bar{\Delta}} u_k - \min_{\bar{\Delta}} u_k \leq C_1$, we have

$$\partial_1 u \leq C_o, \quad |\partial_2 u| \leq C_o \text{ in } \mathcal{D}^\ell$$

for some constant $C_o > 0$ independent of k . Let $p^i = (0, c_i) \in \mathcal{D}^\ell, i = 1, \dots, 5$, be the points with $c_1 < c_2 < c_3 < c_4 < c_5$. By the C^0 -convergence of u_k and $\partial_{22}^2 u \geq C_7$, we have for $i = 1, \dots, 4$

$$\partial_2 u(p^{i+1}) - \partial_2 u(p^i) > \epsilon_o \tag{8.19}$$

for some positive constant ϵ_o independent of k . Let $a_i = \lim_{k \rightarrow \infty} \exp\{\frac{1}{2}\partial_2 u_k(p^i)\}$, $i = 1, \dots, 5$. Without loss of generality we assume that

$$a_1 = \frac{1}{e}, \quad a_2 = \frac{2}{e}, \quad a_3 = 1, \quad a_4 = \frac{e}{2}, \quad a_5 = e$$

(for general case, we use Remark 8.13 and the same argument). Then we have

$$\begin{aligned}\ell_{c_3+\delta_1, c_4-\delta_1} &\subset \tau_{f_k}(J^+), & \ell_{c_4+\delta_1, c_5-\delta_1} &\subset \tau_{f_k}(I^+ \setminus J^+), \\ \ell_{c_1+\delta_1, c_2-\delta_1} &\subset \tau_{f_k}(I^- \setminus J^-)\end{aligned}\quad (8.20)$$

as k large enough, where $\delta_1 = \min_{i=1,\dots,4} \frac{c_{i+1}-c_i}{100}$. Let $\Omega_1 := \tau_f^{-1}(\mathcal{D}^\ell)$.

Now we discuss three cases.

Case 1. Suppose (1) in Lemma 8.10 is true. Set $\tilde{I} = I^+ \setminus J^+$, then

$$\max_{s \in \tilde{I}} H(s) \leq C_{12} \min H. \quad (8.21)$$

Let $\mathfrak{z}_* \in Z$ such that $\tau_f(\mathfrak{z}_*)$ is the middle of ℓ_{c_4, c_5} . Set $q_* = \tau_f(\mathfrak{z}_*)$. By Theorem 6.7 and (8.20) we can find a geodesic ball $B_{2a}(\mathfrak{z}_*)$ such that $B_{2a}(\mathfrak{z}_*) \subset \Omega_1$, where $a > 0$ is a constant independent of k . Since W_g is uniform bounded in Ω_1 , it follows from (8.21) that

$$W(z) \leq N_1 \min_{q \in \tilde{I}} W(q) \leq N_1 W(\mathfrak{z}_*), \quad \forall z \in B_{2a}(\mathfrak{z}_*), \quad (8.22)$$

for some constants $N_1 > 0$ independent of k . It follows from Theorem 4.13 and (8.22) that

$$\mathcal{K} \leq C_1 \text{ in } B_a(\mathfrak{z}_*) \quad (8.23)$$

for some constant $C_1 > 0$ independent of k . By (8.22) and (8.23) we obtain that f satisfies the assumption in Lemma 4.10. Applying Lemma 4.10 with $\alpha = N_1 W(\mathfrak{z}_*)$, we can find a constant $C_2 > 0$ independent of k such that in $B_{a/2}(\mathfrak{z}_*)$,

$$\left[\frac{W}{W(\mathfrak{z}_*)} \right]^{\frac{1}{2}} \Psi \leq C_2.$$

Notice that $\Psi = \|\nabla \log W\|_f^2$. Let $a' = \min(a/2, \frac{1}{2\sqrt{C_2}})$; we have, for any $z \in B_{a'}(\mathfrak{z}_*)$,

$$\frac{1}{2} \leq \left[\frac{W(z)}{W(\mathfrak{z}_*)} \right]^{\frac{1}{4}} \leq \frac{3}{2}. \quad (8.24)$$

On the other hand, by Lemma 8.14, there is a $p' \in \tau_{f_k}(B_{a'}(\mathfrak{z}_*))$ such that $d(p', \partial\Delta) = c'$ for some constant $c' > 0$ independent of k .

We claim that

There is a constant $C > 0$ independent of k , such that $\xi_1(p') \geq C$.

Proof of the Claim. If the Claim is not true, $\lim_{k \rightarrow \infty} \xi_1(p'_k) = 0$. Without loss of generality we can assume that $\partial_2 u_k(p'_k) = 0$ (for general case, since $|\partial_2 u(p')| \leq C$ we can use $u - \partial_2 u(p')\xi_2$ instead of u , and use the same argument). Consider the function $u^* = u - \partial_1 u(p')\xi_1$. Then $u^*(p') = \inf u^*$. By Lemma 7.5 in [9] we have

$$\inf_{\ell} u^* - u^*(p') \geq C_3 > 0 \quad (8.25)$$

for some constant $C_3 > 0$ independent of k . Since $\partial_1 u(p') \leq C_o$, we discuss two subcases. **Subcase 1** $\partial_1 u(p') < 0$. Then

$$\inf_{\ell} u - u(p') \geq \inf_{\ell} u^* - u^*(p') \geq C_3.$$

Subcase 2 $0 \leq \partial_1 u(p') < C_o$. Then by $\lim_{k \rightarrow \infty} \xi_1(p'_k) = 0$ we have

$$\inf_{\ell} u - u(p') \geq \inf_{\ell} u^* - u^*(p') - \frac{C_3}{2} \geq \frac{C_3}{2},$$

as k large enough.

For two cases we have $\inf_{\ell} u - u(p') \geq \frac{C_3}{2}$. By this and Proposition 6.1, we get a contradiction. The Claim is proved.

Let $z = \nabla^u(p') \in B_{a'}(\mathfrak{z}_*)$ be the corresponding point of p' . Following from the Claim and the interior regularity, $W(z)$ is bounded as $C^{-1} < W(z) < C$ for some constant C independent of k . By (8.24), $W(\mathfrak{z}_*)$ is bounded above, therefore $H(\mathfrak{z}_*) > C_5 > 0$ for some constant C_5 independent of k . This violates the assumption that $H_k(\mathfrak{z}_{*k}) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, we get a contradiction when (1) in Lemma 8.10 holds.

Case 2. Suppose that (2) in Lemma 8.10 is true. The proof is the same as Case 1.

Case 3. Now suppose that (3) in Lemma 8.10 is true. For each $(0, s) \in J^+$, let $\mathfrak{z}(s) = (0, s)$, and $p(s), q(s)$ be the image of $r(s), \mathfrak{z}(s)$ via τ_f . If there is a point $(0, s_o) \in J^+$ such that $d(s_o) = 0$, we have

$$\max_I H \leq C_{11} \max_{q \in \Gamma_{s_o}} H(q) = C_{11} H(r(s_o)) \leq C_{11}^2 \min_M H.$$

Then repeating the argument of Case 1 we obtain the Theorem. In the following we assume that for any $(0, s) \in J^+$, $d(s) > 0$. Set $d(s) = d(\ell, p(s))$. Define

$$B(s) = \{\xi \in \ell \mid d_E(\xi, q(s)) \leq d(s)\},$$

and

$$\mathcal{A}(s) = \frac{1}{2d(s)} \int_{B(s)} \partial_{22}^2 u d\xi_2.$$

We need the following lemma.

Lemma 8.15 *There are a constant $\mathcal{C}_{13} > 0$ independent of k and $(0, s) \in J^+$ such that*

$$\mathcal{A}(s) \leq \mathcal{C}_{13}.$$

Proof. By (8.20) we have $\ell_{c_3+\delta_1, c_4-\delta_1} \subset \tau_f(J^+)$. Then $\{B(\gamma), (0, \gamma) \in J^+\}$ covers $\ell_{c_3+\delta_1, c_4-\delta_1}$. By the assumption $\lim_{k \rightarrow \infty} \min_M H_k = 0$ and the interior regularity we have $\lim_{k \rightarrow \infty} \max_{(0, s) \in J^+} d_k(s) = 0$. In fact, if there are a constant $C_0 > 0$ and a subsequence of u_k , still denote by u_k , such that $\max_{s \in J^+} d_k(s) > C_0$, for some constant $C_0 > 0$ independent of k , then as in **Case 1** we have $\min_M H \geq \mathcal{C}_{11} H(r(s)) \geq C_3$ for some constant $C_3 > 0$ independent of k . It contradicts to the assumption $\lim_{k \rightarrow \infty} \min_M H_k = 0$. Then we can assume that for any $\gamma \in J^+$

$$B(\gamma) \subset \ell_{c_3, c_4}.$$

By the compactness of $\ell_{c_3+\delta_1, c_4-\delta_1}$ there are finitely many sets $\{B(\gamma_i)\}_{i=1}^N$ covering $\ell_{c_3+\delta_1, c_4-\delta_1}$, such that for any $1 \leq i < j < k \leq N$

$$B(\gamma_i) \cap B(\gamma_j) \cap B(\gamma_k) = \emptyset.$$

Then we have

$$|c_4 - 2\delta_1 - c_3| < \sum_{i=1}^N |B(\gamma_i)| \leq 2|c_4 - 2\delta_1 - c_3|,$$

and

$$\sum_{i=1}^N \int_{B(\gamma_i)} \partial_{22}^2 u d\xi_2 \leq 2 \int_{\ell_{c_3, c_4}} \partial_{22}^2 u d\xi_2 \leq 2 \left| \frac{\partial u}{\partial \xi_2}(p^4) - \frac{\partial u}{\partial \xi_2}(p^3) \right|.$$

Then there is a point $s \in \{\gamma_i\}_1^N$ such that

$$\frac{\int_{B(s)} \partial_{22}^2 u d\xi_2}{|B(s)|} \leq \frac{2 \left| \frac{\partial u}{\partial \xi_2}(p^4) - \frac{\partial u}{\partial \xi_2}(p^3) \right|}{|c_4 - 2\delta_1 - c_3|}.$$

Since the right hand side is a constant independent of k , we have proven the lemma. ■

Now we focus on $\Gamma(s)$, where $(0, s)$ is provided by Lemma 8.15. There are three special points: two ends $\mathfrak{z}(s)$ and $r(s)$, and $v(s)$, where $v(s)$ is an H -maximal point on $\Gamma(s)$. Let $q(s), p(s), \tilde{p}(s)$ be their images via τ respectively.

It follows from (8.20) and Theorem 6.7 that there are two constants $a, b > 0$ independent of k such that

$$B_a(q(s)) \cup B_a(p(s)) \cup B_a(\tilde{p}(s)) \subset D_b(p(s)), \quad D_b(p(s)) \subset\subset \mathcal{D}^\ell \quad (8.26)$$

if k is large enough. Then by $\min_{\Omega_1} H \geq C_1 H(r(s))$ we have

$$\max_{\Omega_1} W \leq CW(r(s)).$$

Without loss of generality, by translation, we assume that $\xi(q(s)) = 0$. We add a linear function to u such that the minimum point of u is $p(s)$. Such a modification does not affect $\frac{W}{W(r(s))}$. At the mean while, we notice that $q(s)$ becomes the minimum point of u on the boundary (this is due to the fact that $r(s)$ and $\mathfrak{z}(s)$ have the same x_2 coordinate).

Consider the affine transformation (A_1, λ_1) , where

$$A_1(\xi_1, \xi_2) = (\alpha_1 \xi_1, \beta_1 \xi_2), \quad \lambda_1 = \alpha_1 = (\beta_1)^2 = [d(s)]^{-2}.$$

Let $u^\circ, p^\circ(s), \mathcal{D}^{\circ\ell}, \dots$ be the corresponding objects of $u, p(s), \mathcal{D}^\ell, \dots$. Then

$$u^\circ(\xi^\circ) = \lambda_1 u(A_1^{-1} \xi^\circ).$$

Under this affine transformation $B(s)$ is transformed to $(\mathfrak{t}_2^*)_{-1,1} = \{(0, \xi_2^\circ) \mid -1 \leq \xi_2^\circ \leq 1\}$. By a direct calculation, we have

Lemma 8.16 *For u° we have the following facts:*

- (1), $d(p^\circ(s), \mathfrak{t}_2^*) = 1$;
- (2), $W^\circ(z) \leq CW^\circ(r^\circ(s))$ for $z \in \Omega_1^\circ$;
- (3), $\xi_1^\circ \det \left(\frac{\partial u^\circ}{\partial \xi_i^\circ \partial \xi_j^\circ} \right) \geq C_5$;
- (4), $\frac{\partial^2 u^\circ}{\partial (\xi_2^\circ)^2} \geq C_7$ on $(\mathfrak{t}_2^*)_{-c,c}$ where $c = b[d(s)]^{-1}$ and b is the constant in (8.26);
- (5), $|\frac{\partial u^\circ}{\partial \xi_2^\circ}| \leq 2C_{13}$ on $(\mathfrak{t}_2^*)_{-1,1}$.
- (6),

$$\frac{W(r(s))}{W(v(s))} = \frac{W^\circ(r^\circ(s))}{W^\circ(v^\circ(s))}$$

Proof. (1) follows from that $d^2(p^\circ, \mathfrak{t}_2^*) = \lambda_1 d^2(p, \mathfrak{t}_2^*)$. By $W^\circ(z^\circ) = \beta_1^2 W(B_{\mathbb{C}}^{-1} z^\circ)$ and $\max_{\Omega_1} W \leq CW(r(s))$ we obtain (2) and (6), here $B_{\mathbb{C}}$ denote the transformation on \mathbb{C}^2 induced by transformation (A_1, λ_1) . Noting that

$$\det \left(\frac{\partial^2 u^\circ}{\partial \xi_i^\circ \partial \xi_j^\circ} \right) (\xi^\circ) = \frac{\lambda_1^2}{\alpha_1^2 \beta_1^2} \det(\partial_{ij}^2 u)(\xi),$$

from Lemma 5.2 we have (3). (4) follows from Lemma 6.2. Since $\partial_2 u^\circ(0) = 0$, for any $p \in \ell_{-1,1}$ we have

$$\left| \frac{\partial u^\circ}{\partial \xi_2^\circ}(p) \right| \leq \int_{-1}^1 \frac{\partial^2 u^\circ}{\partial (\xi_2^\circ)^2} d\xi_2^\circ = \frac{\int_{B(s)} \partial_{22}^2 u d\xi}{d(s)} \leq 2\mathcal{C}_{13},$$

where the last inequality follows from Lemma 8.15. Then (5) follows. q.e.d.

For every u_k we apply the above affine rescaling to get u_k° . Since $d_k(s_k) \rightarrow 0$ as $k \rightarrow \infty$ (see the argument of the proof of Lemma 8.15), we have

$$\lim_{k \rightarrow \infty} \max_{\mathcal{D}_k^{\circ \ell}} \|\mathcal{S}(u_k^\circ)\|_{C^3} = 0.$$

By Lemma 8.16 we conclude that u_k° satisfies the conditions in Theorem A.2. It follows that u_k° locally uniformly C^∞ -converges to a smooth and strictly convex function u_∞° . In particular, there is a constant $C_0 > 1$ independent of k such that

$$C_0^{-1} \leq W_k^\circ(r_k^\circ(s_k)) \leq C_0. \quad (8.27)$$

Let \tilde{p}_∞° be the limit of $\tilde{p}_k^\circ(s_k)$. If $\tilde{p}_\infty^\circ \notin \mathfrak{t}_2^*$; then by the smooth convergence of u_k° we can find a constant $C'_0 > 1$ independent of k such that

$$(C'_0)^{-1} \leq W_k^\circ(v_k^\circ(s_k)) \leq CW_k^\circ(r_k^\circ(s_k)) \leq C'_0. \quad (8.28)$$

as k large enough.

Now we assume that $\tilde{p}_\infty^\circ \in \mathfrak{t}_2^*$. Noting that Theorem 4.13 holds in $B_a(v(s))$ and $\max_I H \leq C_{11}H(v(s))$, we obtain

$$\mathcal{K} \leq C_1 \frac{\max_{B_a(s)} W}{\min_I W} \leq C_2 \frac{W(r(s))}{W(v(s))} \quad \text{in } B_{a/2}(v(s)). \quad (8.29)$$

Applying Theorem A.2 we have that $B_\epsilon(\tilde{p}^\circ)$ is bounded in a Euclidean ball $D_\delta(\tilde{p}^\circ)$, where ϵ, δ are the constants in Theorem A.2. By (8.27) we obtain that $W^\circ \leq CW^\circ(r^\circ(s)) \leq CC_0$ in $B_{\frac{a}{2d(s)}}(v^\circ(s))$; moreover,

$$\mathcal{K}^\circ \frac{W^\circ(v^\circ(s))}{W^\circ(r^\circ(s))} = \mathcal{K} d^2(s) \frac{W(v(s))}{W(r(s))} \leq C_2 d^2(s), \quad \text{in } B_{\frac{a}{2d(s)}}(v^\circ(s)). \quad (8.30)$$

Since $\lim_{k \rightarrow \infty} d_k(s_k) = 0$, by (8.27) and (8.30) we have

$$\lim_{k \rightarrow \infty} \left[W_k^\circ(v_k^\circ(s_k)) \max_{B_4(v_k^\circ(s_k))} \mathcal{K}_k^\circ \right] = 0.$$

Then by Lemma 4.11, we conclude that $W_k^\circ(v_k^\circ(s_k))$ is bounded below. For both cases we have (8.28). Then back to Theorem 4.13, we have $\mathcal{K}_k \leq C$ for some constant $C > 0$ independent of k . We are now back to the first case. Then we repeat the argument after (8.23) to get a contradiction. This completes the proof of Theorem 8.1. ■

9 Proof of Theorem 1.1

It suffices to prove Theorem 3.7, by Remark 3.9 and Remark 3.10. We already have the interior regularity. It remains to prove the regularity on divisors.

9.1 Regularity on edges

Let ℓ be any edge and $\xi^{(\ell)} \in \ell$ such that $|\mathcal{S}(\xi^{(\ell)})| > 0$. Recall that $f_k = g + \phi_k$, where f_k, g are Legendre transform of u_k, v respectively; and $\phi_k \in C_{\mathbb{T}^2}^\infty(M)$. By Remark 3.9 and Remark 8.13 we can assume that $|\mathcal{S}_k| > \delta > 0$ in

$$\Omega := \{(z_1, z_2) | \log |z_1|^2 \leq \frac{1}{2}, |\log |z_2|^2| \leq 1\}, \quad D_a(\xi^{(\ell)}) \subset \tau_{f_k}(\Omega)$$

and $|z_1(\mathfrak{z}_k^{(\ell)})| = 0, |z_2(\mathfrak{z}_k^{(\ell)})| = 1$, where δ, a are positive constants independent of k , $\mathfrak{z}_k^{(\ell)} \in Z_\ell$ whose image of the moment map is $\xi^{(\ell)}$.

We omit the index k if there is no danger of confusion. By Theorem 6.7, we conclude that there is a constant $\epsilon > 0$ that is independent of k such that $B_\epsilon(\xi^{(\ell)})$ is uniformly bounded in $D_a(\xi^{(\ell)})$. Then $B_\epsilon(\mathfrak{z}^{(\ell)})$ is uniformly bounded. Hence, on this domain, we assume that all data of g_ℓ are bounded.

Then we conclude that on $B_\epsilon(\mathfrak{z}^{(\ell)})$ there is a constant $C > 0$ independent of k such that

1. $C^{-1} \leq W \leq C$: this is due to the bounds on H_f and W_{g_ℓ} on $B_\epsilon(\mathfrak{z}^{(\ell)})$;
2. $\mathcal{K} \leq C$: this follows from Theorem 8.1 and Theorem 4.13.

Hence, by Theorem 4.12, we have the regularity of f on $B_\epsilon(\mathfrak{z}^{(\ell)})$.

9.2 Regularity at vertices

Let ϑ be any vertex. By the results in section 9.1, there is a bounded open set $\Omega_\vartheta \subset \mathbb{U}_\vartheta$, independent of k , such that $\vartheta \in \tau(\Omega_\vartheta)$ and the regularity of f_ϑ holds in a neighborhood of $\partial\Omega_\vartheta$.

We omit the index k if there is no danger of confusion. We now quote a lemma in [9]. Let $f = f_\vartheta$. Recall that $\phi = f - g \in C_{\mathbb{T}^2}^\infty(M)$, where f, g are Legendre transform of u, v respectively. Let

$$T = \sum f^{i\bar{i}}, \quad P = \exp(\kappa W^\alpha) \sqrt{W} \Psi, \quad Q = e^{N_1(|z|^2 - A)} \sqrt{W} T.$$

Then

Lemma 9.1 *Let $\Omega \subset \mathbb{U}_\vartheta$. Suppose that on Ω*

$$\max_{\tau_f(\bar{\Omega})} \left(|\mathcal{S}| + \left| \frac{\partial \mathcal{S}}{\partial \xi_i} \right| \right) \leq C_0, \quad W \leq C_0, \quad |z| \leq C_0 \quad (9.1)$$

for some constant $C_0 > 0$ independent of k . Then we may choose

$$A = C_0^2 + 1, N_1 = 100, \alpha = \frac{1}{3}, \kappa = [4C_0^{\frac{1}{3}}]^{-1} \quad (9.2)$$

such that

$$\square(P + Q + \mathcal{C}_7 f) \geq \mathcal{C}_6 (P + Q)^2 > 0 \quad (9.3)$$

for some positive constants \mathcal{C}_6 and \mathcal{C}_7 that depend only on C_0 and n .

We apply this result to Ω_ϑ . We then have that P and Q are bounded above. By Theorem 8.1 we have W is bounded below and above in Ω_ϑ . It follows that T is bounded above. Therefore we have a constant C such that

$$C^{-1} \leq \nu_1 \leq \nu_2 \leq C,$$

where ν_1 and ν_2 are eigenvalues of $(f_{i\bar{j}})$.

It is then routine to get the regularity on Ω_ϑ . We have proved that ϕ_k uniformly C^∞ -converges to a function $\phi \in C_{\mathbb{T}^2}^\infty(M)$ with $\mathcal{S}(\phi + g) = \mathcal{S} \circ \nabla^f$.

We have finished the proof of Theorem 3.7.

Appendix

A A convergence theorem

Let $u \in \mathcal{C}^\infty(\mathbf{h}^*, v_{\mathbf{h}^*}; K_o)$. On $\mathbf{U}_{\mathbf{h}^*}$, let $z_o = \nabla^u(p_o)$.

Suppose we have a sequence of normalized triples $(u_k, p_{ok}, \check{p}_k)$ (cf. section 4.2) with

$$\lim_{k \rightarrow \infty} \max \|\mathcal{S}(u_k)\|_{C^3(\mathbf{h}^*)} = 0.$$

Definition A.1 Let $N = \max(100, (100 + C)b)$, where b is the constant in Theorem 4.9 and C is the constant in (6.11). A normalized triple (u, p_o, \check{p}) is called a bounded normalized triple if it satisfies

$$W \leq C_1 W(z_o) \text{ in } B_N(z_o); \quad (\text{A.1})$$

$$\det(\partial_{ij}^2 u) \geq (C_1 \xi_1)^{-1}; \quad \text{in } \mathbf{h}^* \quad (\text{A.2})$$

$$\partial_{22}^2 u|_{\mathfrak{t}_2^*} \geq C_1^{-1}; \quad (\text{A.3})$$

$$|\partial_2 u(\xi)| \leq C_1, \quad \forall \xi \in (\mathfrak{t}_2^*)_{-1,1}. \quad (\text{A.4})$$

In this section we prove the following convergence theorem.

Theorem A.2 Let $(u_k, p_{ok}, \check{p}_k)$ be a sequence of bounded normalized triple such that

$$\lim_{k \rightarrow \infty} \max \|\mathcal{S}(u_k)\|_{C^3(\mathbf{h}^*)} = 0.$$

Then

- (1) there is a subsequence u_{k_n} such that p_{ok_n} converges to a point $p_{o\infty}$, and u_{k_n} locally uniformly C^6 -converges to a strictly convex function u_∞ in the interior of \mathbf{h}^* ,
- (2) there exist a subsequence u_{k_n} and two constants $\epsilon, \delta > 0$, independent of k , such that

$$B_\epsilon^{k_n}(\check{p}_{k_n}) \cap \mathbf{h}^* \subset D_\delta(\check{p}_{k_n}) \cap \mathbf{h}^*$$

and in $D_\delta(\check{p}_{k_n}) \cap \mathbf{h}^*$

$$|\partial_2 u_{k_n}| \leq C_{15}, \partial_1 u_{k_n} \leq C_{15}$$

for some constant C_{15} independent of k .

To prove Theorem A.2 we need the following lemmas.

Lemma A.3 *Let $(u_k, p_{ok}, \check{p}_k)$ be a sequence of bounded normalized triple such that*

$$\lim_{k \rightarrow \infty} \max \|\mathcal{S}(u_k)\|_{C^3(\mathbf{h}^*)} = 0.$$

Then there is a constant $\mathcal{C}_{16} > 1$ such that

$$\mathcal{C}_{16}^{-1} < \xi_1(p_{ok}) < \mathcal{C}_{16}.$$

Proof. Let u be a function in the sequence u_k in the Lemma. The proof of $\xi_1(p_{ok}) > \mathcal{C}_{16}^{-1}$ is essentially the same as Lemma 7.9 in [9]. In the following we assume that $\lim_{k \rightarrow \infty} \xi_1(p_{ok}) \rightarrow \infty$. Consider a coordinate transformation

$$\tilde{\xi}_1 = \frac{\xi_1}{a}, \quad \tilde{\xi}_2 = \xi_2 + b\xi_1, \quad \tilde{u}(\tilde{\xi}) = u(a\tilde{\xi}_1, \tilde{\xi}_2 - ab\tilde{\xi}_1), \quad (\text{A.5})$$

where $a = \xi_1(p_o)$. Obviously by the assumption we have $\lim_{k \rightarrow \infty} a_k = +\infty$. Denote by \tilde{p}_o the image of p_o . Choose b such that $\tilde{\xi}_2(\tilde{p}_o) = 0$. Let $\tilde{p}_1 = (0, -1)$, $\tilde{p}_2 = (0, 1)$. Following from (A.4) we have

$$0 \leq \tilde{u}(\tilde{p}_i) \leq C_1, \quad i = 1, 2. \quad (\text{A.6})$$

Take the triangle $\Delta\tilde{p}_1\tilde{p}_2\tilde{p}_o$. By the convexity of \tilde{u}_k , (6.11) and (A.9) we conclude that \tilde{u}_k locally uniformly C^1 -converges to a convex function \tilde{u}_∞ in $\Delta\tilde{p}_1\tilde{p}_2\tilde{p}_o$. Then for any nonempty open set $U \subset \subset \Delta\tilde{p}_1\tilde{p}_2\tilde{p}_o$, we have

$$MA_u(U) = \int_U \det(\partial_{ij}^2 u) d\xi \leq C_3 \quad (\text{A.7})$$

for some constant $C_3 > 0$, as k large enough. Here $MA_u(U)$ denotes the Monge-Ampere measure of U . On the other hand, it follows from $\det(\tilde{\partial}_{ij}^2 \tilde{u})\tilde{\xi}_1 = a \det(\partial_{ij}^2 u)\xi_1$ and (A.2) that

$$\det(\tilde{\partial}_{ij}^2 \tilde{u}) \geq aC_1^{-1}(\tilde{\xi}_1)^{-1}, \quad (\text{A.8})$$

where $\tilde{\partial}_{ij}^2 \tilde{u} = \frac{\partial^2 \tilde{u}}{\partial \tilde{\xi}_i \partial \tilde{\xi}_j}$. Since $\lim_{k \rightarrow \infty} a_k = +\infty$, by (A.8) we have

$$\lim_{k \rightarrow \infty} MA_{u_k}(U) = +\infty.$$

It contradicts to (A.10). The Lemma is proved.

Proof of of Theorem A.2. We omit the index k . We first prove (1). By Lemma A.3 we can assume that $\xi_1(p_o) = 1$. Consider a coordinate transformation

$$\tilde{\xi}_1 = \xi_1, \quad \tilde{\xi}_2 = \xi_2 + a\xi_1, \quad \tilde{u}(\tilde{\xi}) = u(\tilde{\xi}_1, \tilde{\xi}_2 - a\tilde{\xi}_1).$$

Denote by \tilde{p}_o the image of p_o . Choose a such that $\tilde{\xi}_2(\tilde{p}_o) = 0$. Let $\tilde{p}_1 = (0, -1)$, $\tilde{p}_2 = (0, 1)$. Following from (A.4) we have

$$0 \leq \tilde{u}(\tilde{p}_i) \leq C_1, \quad i = 1, 2. \quad (\text{A.9})$$

Take the triangle $\Delta\tilde{p}_1\tilde{p}_2\tilde{p}_o$. By the convexity of \tilde{u}_k , (6.11) and (A.9) we conclude that \tilde{u}_k converges to a convex function \tilde{u}_∞ in $\Delta\tilde{p}_1\tilde{p}_2\tilde{p}_o$. It follows from $\det(\tilde{\partial}_{ij}^2 \tilde{u})\tilde{\xi}_1 = \det(\partial_{ij}^2 u)\xi_1$ and (A.2) that

$$\det(\tilde{\partial}_{ij}^2 \tilde{u}) \geq C_1^{-1}(\tilde{\xi}_1)^{-1}. \quad (\text{A.10})$$

Using Alexandrov-Pogorelov Theorem and the methods of the proof of Theorem 3.6 in [9] we conclude that \tilde{p}_{ok} converges to a point $\tilde{p}_{o\infty}$ and \tilde{u}_k uniformly C^6 -converges to a strictly convex function \tilde{u}_∞ in a neighborhood of $p_{o\infty}$. In particular, there are positive constants C'_1, C'_2 and σ independent of k , such that

$$0 < C'_1 \leq \tilde{W}_k(q) \leq C'_2, \quad \forall q \in \tau_{\tilde{u}_k}^{-1}(S_{\tilde{u}_k}(\tilde{p}_o, \sigma)). \quad (\text{A.11})$$

A direct calculation gives us

$$\exp(-a_k \operatorname{Re} \tilde{z}_2) \tilde{W}_k = W_k, \quad \tilde{\mathcal{K}}_k = \mathcal{K}_k. \quad (\text{A.12})$$

Obviously $\tilde{W}_k(\tilde{z}_{ok}) = W_k(z_{ok})$. Note that

$$C_1 \geq \frac{W(z)}{W(z_o)} = \exp\{-a \operatorname{Re} \tilde{z}_2(\tilde{z})\} \frac{\tilde{W}(\tilde{z})}{\tilde{W}(\tilde{z}_o)} \geq \exp\{-a \operatorname{Re} \tilde{z}_2(\tilde{z})\} \frac{C'_1}{C'_2}. \quad (\text{A.13})$$

where we use (A.11) in the last inequality. From these equations and the convergence we conclude that $|a_k|$ has a uniform upper bound. (1) of the theorem is proved.

Next we prove (2). Take a coordinate translation $\xi^* = \xi - \xi(p_o)$. Then $\xi^*(p_o) = 0$ and

$$\det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) (p) = \det \left(\frac{\partial^2 u}{\partial \xi_i^* \partial \xi_j^*} \right) (p)$$

By the same argument as in section 5.2 (Lemma 5.3, Lemma 5.4 and Lemma 5.5) we obtain that for any $p \in E_{-1,1}^1$

$$\log \det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) (p) \leq C_2'' (1 - \log d_E(p, \mathfrak{t}_2^*)) = C_2'' (1 - \log \xi_1(p)). \quad (\text{A.14})$$

On the other hand

$$\log C_0' \geq \log W(z_o) + \log C_1 \geq \log W = -\log \det(\partial_{ij} u) - \frac{\partial u}{\partial \xi_1},$$

where in the first inequality we use the result in (1). Combining this and (A.14) we have

$$-\frac{\partial u}{\partial \xi_1} \leq C_3 (1 - \log \xi_1).$$

By integration we can obtain for any $\xi_1 \leq 1, |\xi_2| \leq 1$,

$$|u(0, \xi_2) - u(\xi_1, \xi_2)| \leq C_4 \sqrt{\xi_1},$$

where C_4 is a constant independent of k . Using (A.4) we obtain that for any $p, q \in (\mathfrak{t}_2^*)_{-1,1}$

$$|u(p) - u(q)| \leq C_1 |\xi_2(p) - \xi_2(q)|,$$

where C_1 is the constant in (A.4). Then for any constant $\delta \in (0, 1)$, we obtain the C_0 -convergence of u_k on $E_{-1+\delta, 1-\delta}^\delta$. By the same argument as in Proposition 6.6 and Theorem 6.7 we can prove (2). ■

B Construction of Scalar Curvature

Fix a point q_ℓ inside ℓ . We assign a sign $\text{sign}(\ell) = \pm 1$ to q_ℓ . Then we conclude that

Proposition B.1 *There is a potential function v on Δ such that for any ℓ*

$$\text{sign}(\ell) K(q_\ell) > 0,$$

where $K = \mathcal{S}(v)$.

Let ℓ be a edge. We choose a coordinate on \mathfrak{t}^* such that $\ell = \{\xi | \xi_1 = 0\}$ and q_ℓ is the origin. For any $\delta > 0$, let

$$\Omega_\delta^1(\ell) =: \{\xi \in \Delta | \xi_1 \leq \delta\}, \quad \Omega_\delta^2(q_\ell) := \{\xi \in \Delta | |\xi_2| \leq \frac{\delta}{2}\},$$

and $\Omega_\delta(q_\ell) = \Omega_\delta^1(\ell) \cap \Omega_\delta^2(q_\ell)$. Set $\Omega^c(\ell) = \Delta \setminus (\Omega_{2\delta}^1(\ell) \cup \Omega_{2\delta}^2(q_\ell))$.

We need the following lemma.

Lemma B.2 *There is a convex function u_ℓ such that $\text{sign}(\ell)\mathcal{S}(u_\ell) > 0$ in $\Omega_\delta(q_\ell)$; and u_ℓ is linear on $\Omega^c(\ell)$.*

Proof. Consider the convex function $u = \xi_1 \log \xi_1 + \psi$ where ψ is a function of ξ_2 . We compute $\mathcal{S}(u) = -\sum u_{ij}^{ij}$. By a direct calculation, we have

$$\mathcal{S}(u) = -\left(\frac{1}{\psi_{22}}\right)_{22}. \quad (\text{B.1})$$

Set $\Psi = (\psi_{22})^{-1}$. We consider Ψ to be a function of the following form:

$$\Psi = a\xi_2^2 + c. \quad (\text{B.2})$$

Then $\mathcal{S}(u) = -2a$. We may choose a to have the right sign. Now we have

$$\phi := \psi_{22} = \frac{1}{a\xi_2^2 + c}.$$

By choosing c large, ψ_{22} is positive.

Now we construct two convex functions $\alpha(\xi_1)$ and $\beta(\xi_2)$, such that

- $\alpha(\xi_1) = \xi_1 \log \xi_1$ when $\xi_1 \leq \delta$ and is a linear function when $\xi_1 \geq 2\delta$;
- and $\beta(\xi_2) = \psi$ when $|\xi_2| \leq \delta$ and is a linear function when $|\xi_2| \geq 2\delta$, where ψ is the function as above.

We use a cut-off function to modify ϕ to be a non-negative function $\tilde{\phi}$ such that $\tilde{\phi}(\xi_2) = \frac{1}{a\xi_2^2 + c}$ when $\xi_2 \leq \delta$ and vanishes when $\xi_2 \geq 2\delta$. Then β can be constructed from $\tilde{\phi}$ such that $\beta'' = \tilde{\phi}$ and $\beta(0) = \phi(0), \beta'(0) = \phi'(0)$. By the same method we can construct the function $\alpha(\xi_1)$.

Let $u_\ell = \alpha(\xi_1) + \beta(\xi_2)$. Then u_ℓ satisfies the lemma. q.e.d.

Proof of Proposition B.1. It is not hard to choose δ and arrange the coordinate system when we construct ψ such that

$$\Omega_\delta(q_\ell) \subset \bigcap_{\ell' \neq \ell} \Omega^c(\ell'). \quad (\text{B.3})$$

Let $\tilde{u} = \sum_\ell u_\ell$, where u_ℓ is the function in Lemma B.2. Then on $\Omega_\delta(q_\ell)$, $\mathcal{S}(\tilde{u}) = \mathcal{S}(u_\ell)$. Hence, in $\Omega_\delta(q_\ell)$,

$$\text{sign}(q_\ell)\mathcal{S}(\tilde{u}) > 0.$$

However, \tilde{u} is not strictly convex. Let

$$u = \tilde{u} + \epsilon(\xi_1^2 + \xi_2^2).$$

For ϵ small enough, $\mathcal{S}(u)$ is a small perturbation of $\mathcal{S}(\tilde{u}) = \mathcal{S}(u_\ell)$ near q_ℓ . The proposition is then proved. q.e.d.

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